Toeplitz determinants related to a discrete Painlevé II hierarchy

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Based on a joint work with Thomas Chouteau (soon on the ArXiv!)

The object of our study

For any $N \geq 1$, we consider the symbol

$$\varphi^{(N)}[\theta_1, \ldots, \theta_N](z) := \varphi(z) = \mathrm{e}^{w(z)},$$

with

$$w(z) := v(z) + v(z^{-1}) \text{ and } v(z) := \sum_{j=1}^{N} \frac{\theta_j}{j} z^j, \; \theta_j \in \mathbb{R}, \; z \in S^1.$$

We focus on the study of the Toeplitz determinants

$$\boxed{D_n := \det(T_n(\varphi))}$$

with $T_n(\varphi)$ being the n-th Toeplitz matrix associated to the symbol $\varphi(z)$

$$T_n(\varphi)_{i,j} := \varphi_{i-j}, \quad i, j = 0, \ldots, n$$

where for every $k \in \mathbb{Z}$, φ_k is the k-th Fourier coefficient of $\varphi(z)$, namely

$$\varphi_k = \int_{-\pi}^{\pi} e^{-ik\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi}, \text{ so that } \sum_{k \in \mathbb{Z}} \varphi_k z^k = \varphi(z).$$

Outline

From random permutations...

Consider the set of permutations S_M with uniform distribution, so that for any $\pi_M \in S_M$ we have

$$\mathbb{P}(\pi_M) = \frac{1}{M!}$$

and denote $\ell(\pi_M)$ the length of the longest increasing sub-sequence of π_M.

Example $\pi_5 = \begin{pmatrix} 4 & 3 & 1 & 2 & 5 \end{pmatrix}$ and $\ell(\pi_5) = 3$.

Ulam problem (1961)

Describe the behavior of $\ell(\pi_M)$ for $M \to \infty$.

Theorem (Baik–Deift–Johansson (1999))

$$\lim_{M\to\infty} \mathbb{P}\left(\frac{\ell(\pi_M) - 2\sqrt{M}}{M^{1/6}} \leq t \right) = F(t), \quad \textit{with} \;\; F(t) := \det\left(1 - \mathcal{K}_{\mathrm{Ai}}\big|_{(t,\infty)}\right).$$

...to random partitions

Theorem (Robinson–Schensted correspondence)

$$RS : \pi_M \ni S_M \to RS(\pi_M) \in \{(P,Q) \in SYT_M \times SYT_M,\ sh(P) = sh(Q)\}$$

is a bijection.

Example Consider the permutation $\pi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix}$.

The RS correspondence associate the pair $RS(\pi_5) = (P_5, Q_5)$

$$\emptyset,\emptyset \to \begin{array}{c}4\end{array},\begin{array}{c}1\end{array} \to \begin{array}{c}3\\4\end{array},\begin{array}{c}1\\2\end{array} \to \begin{array}{c}1\\3\\4\end{array},\begin{array}{c}1\\2\\3\end{array} \to \begin{array}{cc}1&2\\3&\\4&\end{array},\begin{array}{cc}1&4\\2&\\3&\end{array} \to \begin{array}{ccc}1&2&5\\3&&\\4&&\end{array},\begin{array}{ccc}1&4&5\\2&&\\3&&\end{array} = (P_5, Q_5).$$

$$\begin{array}{ccc}1&2&5\\3&&\\4&&\end{array},\begin{array}{ccc}1&4&5\\2&&\\3&&\end{array} \to \begin{array}{cc}1&2\\3&\\4&\end{array},\begin{array}{cc}1&4\\2&\\3&\end{array} \to \begin{array}{c}1\\3\\4\end{array},\begin{array}{c}1\\2\\3\end{array} \to \begin{array}{c}3\\4\end{array},\begin{array}{c}1\\2\end{array} \to \begin{array}{c}4\end{array},\begin{array}{c}1\end{array} \to \emptyset,\emptyset.$$

$x_5 = 5$ $\quad x_4 = 2$ $\quad x_3 = 1$ $\quad x_2 = 3$ $\quad x_1 = 4 \implies SR(P_5 Q_5) = \pi_5.$

The Schensted theorem

In particular, if $\lambda(\pi_M) = (\lambda_1(\pi_M) \geq \lambda_2(\pi_M) \dots)$ is the partition coinciding with $\mathrm{sh}(P) = \mathrm{sh}(Q)$ in the above correspondence, the Schensted theorem (1961) says

$$\ell(\pi_M) = \lambda_1(\pi_M).$$

Example For π_5 we had $\lambda(\pi_5) = (3 \geq 1 \geq 1)$.

Moreover, the uniform distribution on S_M pushes forward through *RS* onto the set $\mathcal{Y}_M$ of all partitions of *M* inducing on it the *Plancherel* measure

$$\mathbb{P}_{\mathrm{Pl.}}(\lambda) = \frac{F_\lambda^2}{M!}, \text{ with } F_\lambda = \#\{P \in \mathrm{SYT}_M,\ \mathrm{sh}(P) = \lambda\}.$$

In this sense

$$\mathbb{P}(\ell(\pi_M) \leq n) = \mathbb{P}_{\mathrm{Pl.}}(\lambda_1(\pi_M) \leq n).$$

Poissonized Plancherel measure and Gessel's theorem

On the set of all partitions of any size $\mathcal{Y} = \bigcup_M \mathcal{Y}_M$ one can consider a *Poissonized* version of the Plancherel measure

$$\mathbb{P}_{\text{P.Pl.}}(\lambda) = \mathrm{e}^{-\theta^2} \left(\frac{\theta^{|\lambda|} F_\lambda}{|\lambda|!} \right)^2, \text{ where } |\lambda| = \text{weight}(\lambda).$$

Theorem (Gessel's formula)

$$\mathbb{P}_{P.Pl.}(\lambda_1 \leq n) = \mathrm{e}^{-\theta^2} D_{n-1}(\varphi)$$

where $\varphi = \varphi^{(1)}[\theta_1 = \theta](z)$.

Remark 1 The result of B–D–J was obtained after the study of asymptotic behavior of these Toeplitz determinants using the RH approach for OPUC and de Poissonization procedure.
Remark 2 This quantity has also a Fredholm determinant representation in terms of a discrete version of the Bessel kernel $\rightarrow$ Giulio's talk!

Borodin's formula

Theorem (Many authors)

For every $n \geq 1$ we have

$$\frac{D_n D_{n-2}}{D_{n-1}^2} = 1 - x_n^2$$

where x_n solves the so called discrete Painlevé II equation, which corresponds to the second order nonlinear difference equation

$$\boxed{\theta(x_{n+1} + x_{n-1})(1 - x_n^2) + n x_n = 0}$$

with initial conditions $x_0 = -1, x_1 = \varphi_1/\varphi_0$.

Remark 1 This result as stated above was proved in different ways from different authors Borodin, Baik, Adler–Van Moerbeke more or less at the same time (2001).
Remark 2 Similar discrete equations appeared previously in Periwal-Schewitz (1990) in the study of some unitary matrix integrals.

The continuous limit

Recall the B–D–J result in this context: $\lim_{\theta\to\infty} \mathbb{P}\left(\frac{\lambda_1 - 2\theta}{\theta^{1/3}} \le t\right) = F(t)$.
Thus by performing the following scaling

$$n = t\theta^{1/3} + 2\theta \quad \Longleftrightarrow \quad t = (n - 2\theta)\theta^{-1/3}.$$

in the formula for the Toeplitz determinants and take the limit for $\theta \to \infty$

$$\frac{D_n D_{n-2}}{D_{n-1}^2} - 1 \quad = \quad -x_n^2, \qquad\qquad x_{n+1} + x_{n-1} = -\frac{n x_n}{\theta(1 - x_n^2)}$$

B–D–J $\downarrow$ $\boxed{x_n = (-1)^n \theta^{-1/3} u(t)}$ $\qquad\qquad$ $\downarrow$ $\boxed{x_n = (-1)^n \theta^{-1/3} u(t)}$

$$\partial_t^2 \log F(t) = -u^2(t), \qquad\qquad \underbrace{u''(t) = 2u^3(t) + tu(t)}_{\text{Painlevé II equation}}$$

which recovers the Tracy-Widom formula (1994) for GUE.

Outline

Multicritical random partitions

(Okunkov, 2001) Consider now on the set of all partitions the *Schur* measures

$$\mathbb{P}_{\text{Sc.}}(\lambda) = Z^{-1} s_\lambda\left[\theta_1, \dots, \theta_N\right]^2,$$

where s_λ can be computed as

$$s_\lambda\left[\theta_1, \dots, \theta_N\right] = \det_{i,j} h_{\lambda_i - i + j}\left[\theta_1, \dots, \theta_N\right], \text{ with } \sum_{k\geq 0} h_k z^k = \mathrm{e}^{V(z)}, \text{ and } Z = \mathrm{e}^{\sum_{i=1}^N \frac{\theta_i^2}{i}}.$$

Remark For $N = 1$ with $\mathbb{P}_{\text{P.Pl.}}(\lambda) = \mathbb{P}_{\text{Sc.}}(\lambda)$ with $\theta_1 = \theta$.

Theorem (Betea–Bouttier–Walsh (2021))

Let $\theta_i = (-1)^{i+1}\frac{(N-1)!(N+1)!}{(N-i)!(N+i)!}\theta = (-1)^{i+1}\hat{\theta}_i$. *Then*

$$\lim_{\theta\to\infty} \mathbb{P}_{Sc.}\left(\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2N+1}}} < t\right) = F_N(t), \quad \textit{with} \quad F_N(t) = \det(1 - \mathcal{K}_{\mathrm{Ai}_{2N+1}}|_{(t,\infty)})$$

where $b = \frac{N+1}{N}, d = \binom{2N}{N-1}^{-1}$.

Higher order Tracy-Widom formula

Theorem (Cafasso–Claeys–Girotti (2019))

For every $N \geq 1$

$$\partial_t^2 \log F_N(t) = -u^2((-1)^{N+1} t)$$

where $u((-1)^{N+1} t)$ *is a particular solution of the* N*-th equation of the Painlevé II hierarchy.*

For $N = 1, 2, 3$ this means that $u(t)$ solves the following ODEs

$$\begin{aligned}
N = 1: &\qquad u'' - 2u^3 = tu, \\
N = 2: &\qquad u'''' - 10u(u')^2 - 10u^2 u'' + 6u^5 = -tu, \\
N = 3: &\qquad u'''''' - 14u^2 u'''' - 56uu'u''' - 70(u')^2 u'' - 42u(u'')^2 + 70u^4 u'' \\
&\qquad + 140u^3 (u')^2 - 20u^7 = tu
\end{aligned}$$

where $'$ denotes the derivative w.r.t. t.

What is left to complete the picture

Discrete

N=1 $\mathbb{P}_{\text{P.Pl.}}(\lambda_1 \leq n) = \mathrm{e}^{-\theta^2} D_{n-1}(\varphi)$ with $\varphi = \varphi^{(1)}[\theta_1 = \theta](z)$ and

$$D_{n-2}D_n/D_{n-1}^2 = 1 - x_n^2$$

with x_n solving

$$\text{dPII}: \ \theta(x_{n+1}+x_{n-1})(1-x_n^2)+nx_n = 0.$$

Continuous

$$\lim_{\theta\to\infty} \mathbb{P}_{\text{P.Pl.}}\left(\frac{\lambda_1 - 2\theta}{\theta^{\frac{1}{3}}} \leq t\right) = F(t)$$

and

$$\partial_t^2 \log F(t) = -u^2(t)$$

with u solving

$$\text{PII}: \quad u''(t) = 2u^3(t) + tu(t).$$

N>1 $\mathbb{P}_{\text{Sc.}}(\lambda_1 \leq n) = \mathrm{e}^{-\sum_i^N \hat{\theta}_i^2/i} D_{n-1}(\varphi)$ with $\varphi = \varphi^{(N)}\left[\hat{\theta}_1, \dots, \hat{\theta}_N\right](z)$ and

what is the recursion relation for D_n?

$\lim_{\theta\to\infty} \mathbb{P}_{\text{Sc.}}\left(\frac{\lambda_1 - b\theta}{(d\theta)^{\frac{1}{2N+1}}} \leq t\right) = F_N(t)$ and $\partial_t^2 \log F_N(t) = -u^2((-1)^{N+1}t)$ with u solving the N-th higher order analogue of PII.

The family of OPUC of interest

We consider the measure for $z = \mathrm{e}^{i\alpha} \in S^1$ given by

$$\mathrm{d}\mu(\alpha) = \varphi(\mathrm{e}^{i\alpha})\frac{\mathrm{d}\alpha}{2\pi} = \mathrm{e}^{w(\mathrm{e}^{i\alpha})}\frac{\mathrm{d}\alpha}{2\pi}.$$

The family $\{p_n(z)\}_{n\in\mathbb{N}}$ of orthogonal polynomials on the unitary circle (OPUC) w.r.t. the measure is given by

$$p_n(z) = \kappa_n z^n + \dots \kappa_0, \quad \kappa_n > 0$$

such that the following relation holds for any index k, h

$$\int_{-\pi}^{\pi} \overline{p_k(e^{i\alpha})} p_h(e^{i\alpha}) \frac{\mathrm{d}\mu(\alpha)}{2\pi} = \delta_{k,h}.$$

The analogue monic orthogonal polynomials $\pi_n(z)$ are $p_n(z) = \kappa_n \pi_n(z)$.

Relation with the Toeplitz determinants

A very well known formula of $p_n(z)$ in terms of the Toeplitz determinants D_n gives

$$p_n(z) = \frac{1}{\sqrt{D_n D_{n-1}}} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-n+1} & \varphi_{-n} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-n+2} & \varphi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \cdots & \varphi_0 & \varphi_{-1} \\ 1 & z & \dots & z^{n-1} & z^n \end{pmatrix}, \quad n \geq 1,$$

from which in particular one deduces that the leading coefficient of $p_n(z)$ is related to the ratio of consecutive Toeplitz determinants as

$$\boxed{\frac{D_{n-1}}{D_n} = \kappa_n^2}$$

R–H problem associated to OPUC

For any fixed $n \geq 0$, the function $Y(z) := Y(z, n; \theta_i) : \mathbb{C} \to \mathrm{GL}(2, \mathbb{C})$ has the following properties

(1) $Y(z)$ is analytic for every $z \in \mathbb{C} \setminus S^1$;

(2) $Y(z)$ has continuous boundary values $Y_\pm(z)$ are related for all $z \in S^1$ through

$$Y_+(z) = Y_-(z) J_Y(z), \quad \text{with} \quad J_Y(z) = \begin{pmatrix} 1 & z^{-n} e^{w(z)} \\ 0 & 1 \end{pmatrix};$$

(3) $Y(z)$ is normalized at ∞ as

$$Y(z) \sim \left(I + \sum_{j=1}^{\infty} \frac{Y_j(n, \theta_i)}{z^j} \right) z^{n\sigma_3}, \quad z \to \infty,$$

where σ_3 denotes the Pauli's matrix $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Solution of this R–H problem

Theorem (Baik–Deift–Johansson (1999))

The R–H problem admits a unique solution $Y(z)$ written as

$$Y(z) = \begin{pmatrix} \pi_n(z) & \mathcal{C}\left(y^{-n}\pi_n(y)\mathrm{e}^{w(y)}\right)(z) \\ -\kappa_{n-1}^2\pi_{n-1}^*(z) & -\kappa_{n-1}^2\mathcal{C}\left(y^{-n}\pi_{n-1}^*(y)\mathrm{e}^{w(y)}\right)(z) \end{pmatrix},$$

where $\pi_{n-1}^*(z)$ is defined as the polynomial of the same degree of $\pi_{n-1}(z)$ such that $\pi_{n-1}^*(z) := z^n\overline{\pi_{n-1}\left(\bar{z}^{-1}\right)}$. and $(\mathcal{C}f(y))(z)$ is the Cauchy transform of f

$$\left(\mathcal{C}f(y)\right)(z) := \frac{1}{2\pi i}\int_{S^1}\frac{f(y)}{y-z}\mathrm{d}y.$$

Moreover, $\det(Y(z)) \equiv 1$.

Remark This is an extension of the R–H approach to orthogonal polynomials on the real line formulated by Fokas–Its–Kitaev (1991).

Important symmetry property

For every fixed $n \geq 0$, the unique solution $Y(z)$ of the R–H problem satisfies

$$Y(z) = \sigma_3 Y(0)^{-1} Y(z^{-1}) z^{n\sigma_3} \sigma_3,$$

where the matrix $Y(0) = Y(0, n; \theta_i)$ is given for any $n \geq 1$ by

$$Y(0, n; \theta_i) = \begin{pmatrix} x_n & \kappa_n^{-2} \\ -\kappa_{n-1}^2 & x_n \end{pmatrix},$$

with $x_n := \pi_n(0)$ and κ_n is the leading coefficient of $p_n(z)$.

Remark Since $\det Y(0, n; \theta_i) = 1$ for every $n \geq 1$ we have

$$\boxed{\frac{\kappa_{n-1}^2}{\kappa_n^2} = 1 - x_n^2}$$

Corollary (Formula for the Toeplitz determinants)

For every $n \geq 1$ we have $\frac{D_{n-2} D_n}{D_{n-1}^2} = 1 - x_n^2.$

Transformation of the R–H problem

Now the function

$$\Psi(z,n;\theta_i) := \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(z,n;\theta_j) \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} \mathrm{e}^{w(z)\frac{\sigma_3}{2}}.$$

satisfies the following properties :

(1) $\Psi(z)$ is analytic for every $z \in \mathbb{C} \setminus \{S^1 \cup \{0\}\}$;

(2) $\Psi(z)$ has continuous boundary values $\Psi_\pm(z)$ related for all $z \in S^1$ by

$$\Psi_+(z) = \Psi_-(z) J_0, \quad J_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(3) $\Psi(z)$ has asymptotic behavior near 0 given by

$$\Psi(z) \sim \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(0) \left(I + \sum_{j=1}^{\infty} z^j \widetilde{Y}_j(n) \right) \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} \mathrm{e}^{w(z)\frac{\sigma_3}{2}}, \quad z \to 0.$$

(4) $\Psi(z)$ has asymptotic behavior near ∞ given by

$$\Psi(z) \sim \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} \left(I + \sum_{j=1}^{\infty} \frac{Y_j(n)}{z^j} \right) \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix} \mathrm{e}^{w(z)\frac{\sigma_3}{2}}, \quad |z| \to \infty.$$

System of equations for Ψ

Proposition

$$\boxed{\Psi(z,n+1) = U(z,n)\Psi(z,n), \quad \partial_z \Psi(z,n) = T(z,n)\Psi(z,n)}$$

with

$$U(z,n) := \begin{pmatrix} z + x_n x_{n+1} & -x_{n+1} \\ -(1-x_{n+1}^2)x_n & 1 - x_{n+1}^2 \end{pmatrix} = \sigma_+ z + U_0(n),$$

$$T(z,n) := T_1(n)z^{N-1} + T_2(n)z^{N-2} + \ldots + T_{2N+1}(n)z^{-N-1} = \sum_{k=1}^{2N+1} T_k z^{N-k},$$

where $T_1(n) = \frac{\theta_N}{2}\sigma_3$.

Remark The symmetry property of $Y(z)$ gives in turn for $T(z,n)$ that

$$T(z^{-1},n) = -z^2\left(K(n)T(z,n)K(n)^{-1} - nz^{-1}I_2\right)$$

with $K(n) := \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(0,n)\sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix}$,

which in terms of $T_j(n)$ reads as $\longrightarrow$

$$T_j(n) = -K(n)T_{2N+2-j}(n)K(n)^{-1},$$

for $j = 1, \ldots, N$ and

$$T_{N+1}(n) = -K(n)T_{N+1}(n)K(n)^{-1} + nI_2.$$

Exploiting the compatibility condition

The compatibility condition of the system for Ψ gives

$$\sigma_+ = T(n+1,z)U(n,z) - U(n,z)T(n,z)$$

with $T(n,z)$ satisfying the prescribed symmetry

$$\downarrow$$

looking at the coefficients of each power z^k for $k = N, \ldots, -1$

$$\downarrow$$

we find

- A system of discrete (in n) equations for $T_k^{ij}(n), i,j \in \{1,2\}$ for $k = 1, \ldots, N+1$ that determines all of them in terms of $x_{n\pm j}, j = -N, \ldots, N$ recursively (on k) starting from the initial condition for $T_1(n)$.
- Plugging the form obtained for the last coefficient $T_{N+1}(n)$ into the equation for $T_{N+1}(n)$ given from the symmetry, a nonlinear $2N$ order discrete equation for x_n which is the N-th equation of the discrete Painlevé II hierarchy.

The case $N = 1$

The Lax matrix $T(z, n)$ results as follows

$$T(n,z) = \frac{\theta_1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} n & -\theta_1 x_{n+1} \\ -\theta_1 v_n x_{n-1} & 0 \end{pmatrix} + \frac{\theta_1}{z^2}\begin{pmatrix} \frac{1}{2} - x_n^2 & x_n \\ v_n x_n & x_n^2 - \frac{1}{2} \end{pmatrix}$$

where $v_n = 1 - x_n^2$ and (as already known from Borodin's formula) we have that x_n satisfies the discrete Painlevé II equation

$$x_{n+1} + x_{n-1} = \frac{n x_n}{\theta_1 (x_n^2 - 1)}.$$

Remark The Lax pair obtained here however is different from the one obtained by Borodin.

The case $N = 2$

The Lax matrix $T(z,n)$ results as follows

$$T(z,n) = z\frac{\theta_2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}\frac{\theta_1}{2} & -\theta_2 x_{n+1}\\ -\theta_2 x_{n-1} v_n & -\frac{\theta_1}{2}\end{pmatrix} +$$

$$\frac{1}{z}\begin{pmatrix} n - \theta_2 x_{n-1}x_{n+1}v_n & -\theta_1 x_{n+1} - \theta_2(v_{n+1}x_{n+2} - x_n x_{n+1}^2)\\ \left(-\theta_1 x_{n-1} - \theta_2(v_{n-1}x_{n-2} - x_n x_{n-1}^2)\right) v_n & \theta_2 x_{n-1}x_{n+1}v_n \end{pmatrix}$$

$$+ \frac{1}{z^2}\begin{pmatrix} -\theta_2 v_n(x_n x_{n-1} + x_n x_{n+1}) + \frac{\theta_1}{2}(v_n - x_n^2) & -\theta_2(v_n x_{n-1} + x_n^2 x_{n+1})\\ -\theta_2(v_n x_{n+1} + x_n^2 x_{n-1})v_n & \theta_2 v_n(x_n x_{n-1} + x_n x_{n+1}) - \frac{\theta_1}{2}(v_n - x_n^2) \end{pmatrix}$$

$$+ \frac{\theta_2}{z^3}\begin{pmatrix} \frac{1}{2} - x_n^2 & x_n\\ v_n x_n & x_n^2 - \frac{1}{2} \end{pmatrix}.$$

and this time x_n satisfies the 4th order nonlinear discrete equation

$$n x_n + \theta_1 v_n (x_{n+1} + x_{n-1}) + \theta_2 v_n \left(x_{n+2}v_{n+1} + x_{n-2}v_{n-1} - x_n(x_{n+1} + x_{n-1})^2\right) = 0,$$

which is known as the second member of the discrete Painlevé II hiearchy.

Final statement

Theorem (T. – Chouteau)

For any fixed $N \geq 1$, for the Toeplitz determinants $D_n, n \geq 1$, we have

$$\frac{D_n D_{n-2}}{D_{n-1}^2} = 1 - x_n^2$$

where now x_n solves the $2N$ order nonlinear difference equation

$$nx_n + \left(v_n + v_n Perm_n - 2x_n \Delta^{-1}\left(x_n - (\Delta + I)x_n Perm_n\right)\right) L^N(0) = 0$$

where L is a discrete recursion operator that acts as follows

$$L(u_n) := \left(x_{n+1}\left(2\Delta^{-1} + I\right)\left((\Delta + I)\, x_n Perm_n - x_n\right) + v_{n+1}\left(\Delta + I\right) - x_n x_{n+1}\right) u_,$$

and $L(0) = \theta_N x_{n+1}$. Here $v_n := 1 - x_n^2$, Δ denotes the difference operator $\Delta : u_n \to u_{n+1} - u_n$ and $Perm_n$ is the transformation

$$\begin{array}{rccc} Perm_n : & \mathbb{C}\left[(x_j)_{j \in [[0,2n]]}\right] & \longrightarrow & \mathbb{C}\left[(x_j)_{j \in [[0,2n]]}\right] \\ & P\left((x_{n+j})_{-n \leqslant j \leqslant n}\right) & \longmapsto & P\left((x_{n-j})_{-n \leqslant j \leqslant n}\right). \end{array}$$

Connection with the Cresswell–Joshi Lax pair

Cresswell and Joshi (1998) introduced the discrete Painlevé II hierarchy as the compatibility condition of the system

$$\Phi(z, n+1) = L(z, n)\Phi(z, n)$$
$$\frac{\partial}{\partial z}\Phi(z, n) = M(z, n)\Phi(z, n)$$

where $L(z, n) := \begin{pmatrix} z & x_n \\ x_n & 1/z \end{pmatrix}$ and $M(z, n) := \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & -A_n(z) \end{pmatrix}$ with A_n, B_n and C_n written in integer powers of z (from z^N to z^{-N}).

Proposition (T.–Chouteau)

The C–J Lax pair and the one found in our work are related through

$$\Phi(z, n) := \sigma_3 \begin{pmatrix} z^{-n+3/2} & 0 \\ 0 & z^{-n+1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_{n-1} & 1 \end{pmatrix} \Psi(z^2, n-1).$$

The other continuous limits

Recall the B–B–W result for $N=2$: it says $\lim_{\theta\to\infty}\mathbb{P}_{\text{Sc.}}\left(\frac{\lambda_1-\frac{3}{2}\theta}{(4^{-1}\theta)^{1/5}}\leq t\right)=F_2(t)$.
Thus by performing the following scaling

$$n=t\left(\frac{\theta}{4}\right)^{1/5}+\frac{3}{2}\theta \quad\Longleftrightarrow\quad t=\left(n-\frac{3}{2}\theta\right)\theta^{-\frac{1}{5}}4^{\frac{1}{5}}.$$

in the formula for the Toeplitz determinants and taking the limit for $\theta\to\infty$

$$\frac{D_nD_{n-2}}{D_{n-1}^2}-1 = -x_n^2,$$

$$nx_n+\theta_1 v_n\left(x_{n+1}+x_{n-1}\right) +\theta_2 v_n\left(x_{n+2}v_{n+1}+x_{n-2}v_{n-1}-x_n(x_{n+1}+x_{n-1})^2\right)=0$$

B–B–W $\downarrow$ $x_n=(-1)^n\left(\frac{\theta}{4}\right)^{-1/5}u(t)$

$\downarrow$ $x_n=(-1)^n\left(\frac{\theta}{4}\right)^{-1/5}u(t)\ \theta_1=\theta,\ \theta_2=\frac{\theta}{4}$

$$\partial_t^2\log F_2(t)=-u^2(t),$$

$$\underbrace{u''''-10u(u')^2-10u^2u''+6u^5=-tu}_{\text{2nd eq. of the Painlevé II hierarchy}}$$

which recovers the result from Cafasso–Claeys–Girotti for $N=2$.

Remark For $N=3$ it works as well by taking $x_n=(-1)^n\left(\frac{\theta}{15}\right)^{-1/7}u(t)$, with $t=\left(n-\frac{3}{4}\theta\right)\theta^{-1/7}15^{1/7}$ and $\theta_1=\theta,\theta_2=\frac{2\theta}{5},\theta_3=\frac{\theta}{15}$.

Open problems

- Give a formal proof for any N that the N-th equation of the discrete Painlevé II hierarchy converges to the N-th equation of the classical Painlevé II hierarchy.
- Understand at the level of the difference between the recursion operators used here and the ones used in Cresswell–Joshi.

Thank you!