# Integrabilità delle densità di Janossy del processo puntuale di Airy *thinned*

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# Outline

# G(aussian) U(nitary) E(nsemble)

GUE is the random matrix ensemble obtained by taking on the set of Hermitian matrices of size $N$ the probability distribution

$$\mathbb{P}(M)dM = \frac{1}{Z_N} e^{-\operatorname{tr} M^2/2} dM.$$

**Remark** Square matrices of size $N$ such that $M = \frac{X+X^\dagger}{2}$ where $X_{h,k} \sim \mathcal{N}(0,1) + i\mathcal{N}(0,1)$, are i.i.d. random variables with

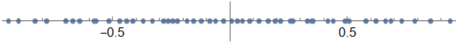
$$\mathcal{N}(0,1) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \quad \text{Gaussian distribution}$$

are distributed as in GUE.

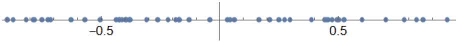
**Problem** Study of the eigenvalues and their limiting behavior for $N \to \infty$.

$$\downarrow$$

Point processes on $\mathbb{R}$.




Eigenvalues of a GUE matrix (scaled by $2\sqrt{N}$), $N = 50$.




$N = 50$ real points uniformly distributed on $[-1, 1]$.

# Global behavior of eigenvalues

Eigenvalues of GUE behave as a determinantal point process (on $\mathbb{R}$).

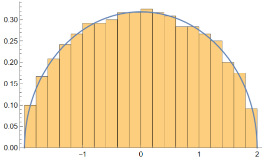
$$\downarrow$$

This means, for every $k \leq N$ Borel subsets $A_j \subset \mathbb{R}, j = 1, \dots, k$ pairwise disjoint, the expected number of distinct eigenvalues $(\lambda_1, \dots, \lambda_k)$ such that $\lambda_i \in A_i$ is given by

$$m_k(A_1 \times \cdots \times A_k) = \int_{A_1} \cdots \int_{A_k} \underbrace{\det_{1 \leq i,j \leq k} K_N(x_i, x_j)}_{=\rho_k^{(N)}(x_1,\dots,x_k) \text{ correlation function}} dx_1 \dots dx_k.$$

where the correlation kernel $K_N$ is explicitly written in terms of Hermite polynomials.

**Remark** The scaled density of GUE eigenvalues $\rho_1^{(N)}(x)/N$ for $N \to +\infty$ converges to the Wigner semicircle law.




Histogram of GUE eigenvalues (scaled by $\sqrt{N}$, $N = 600$) and the Wigner semicircle law.

# Large $N$ limit

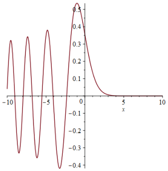
[Gaudin - Metha, Forrester 1993] For $N \to \infty$ the correlation kernel of the eigenvalues of GUE matrices near the edge of the spectrum behaves like

$$N^{-2/3} K_N \left(2 + \frac{x}{N^{2/3}}, 2 + \frac{y}{N^{2/3}}\right) \to \underbrace{K^{\mathrm{Ai}}(x,y)}_{\text{Airy kernel}} = \int_0^{+\infty} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\mathrm{d}u.$$

**Remark** $\mathrm{Ai}(x)$ is a real solution of the Airy equation

$$v''(x) = xv(x)$$

which can be written as $\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^{+\infty} \cos\left(\frac{t^3}{3} + xt\right) dt$.




Plot of $\mathrm{Ai}(x)$.

$$\downarrow$$

[Soshnikov, 2001] $K^{\mathrm{Ai}}(x,y)$ uniquely define the Airy DPP $\mathbb{P}^{\mathrm{Ai}}$ in which

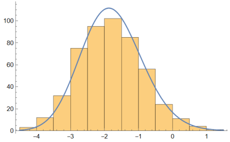
- every configuration has a. s. an infinite number of points;
- every configuration has a. s. a largest point.

$\mathbb{P}^{\mathrm{Ai}}$ is universal $\rightsquigarrow$ free fermions, random partitions and permutations, percolation models.

# Tracy-Widom distribution

[Gaudin - Metha, Forrester 1993] The limiting behavior of the largest eigenvalue of GUE is described by the Tracy-Widom distribution

$$\lim_{N\to\infty} \mathbb{P}\left(\left(\lambda_{max}^{\text{GUE}} - \sqrt{2N}\right)\left(\sqrt{2}N^{1/6}\right) \leq s\right) = F(s),$$



TW distribution and 500 samples of GUE largest eigenvalues (scaled by $\sqrt{N}$, $N = 20$).

which is the Fredholm determinant of the integral operator $\mathcal{K}^{\mathrm{Ai}}$ associated to $K^{\mathrm{Ai}}(x, y)$

$$\begin{aligned} F(s) &= \det\left(1 - \mathcal{K}^{\mathrm{Ai}}|_{(s,+\infty)}\right) \\ &= 1 + \sum_{n\geq 1}^{\infty} \frac{(-1)^n}{n!} \int_{(s,\infty)^n} \det_{i,j=1,\dots n} K^{\mathrm{Ai}}(x_i, x_j) dx_1 \dots dx_n. \end{aligned}$$

**Remark** For every $s \in \mathbb{R}$, $F(s)$ corresponds to the so called gap probability associated to $(s, +\infty)$ : the probability of finding no particle in $(s, +\infty)$ in the process $\mathbb{P}^{\mathrm{Ai}}$.

# Tracy-Widom formula

[Tracy - Widom, 1994] $F(s)$ satisfies the formula

$$\frac{d^2}{ds^2}\log F(s) = -q^2(s)$$

where $q$ is the Hastings-McLeod solution of the Painlevé II equation, i.e. the unique solution of the boundary value problem

$$q''(s) = sq(s) + 2q^3(s)$$

with $q(s) \sim \mathrm{Ai}(s)$ for $s \to +\infty$.

Integrating, we have

$$F(s) = \exp\left(-\int_s^{+\infty}(r-s)q^2(r)dr\right).$$

**Remark** [Picard, 1889 - Painlevé, 1900 - Gambier, 1910] Painlevé equations are 6 nonlinear ODEs of which solutions do not have *movable branch points* and they do not admit generically solutions in terms of known special functions $\rightsquigarrow$ Painlevé transcendents.

# Classical solutions of Korteweg-de Vries

The KdV equation is the integrable PDE given by

$$\partial_t v + \frac{1}{6}\,\partial_x^3 v + 2\,v\,\partial_x v \;=\; 0.$$

**Remark** Every solution $v(x,t)$ corresponds to a solution $U(X,T)$ of the cylindrical KdV equation

$$\partial_T U + \frac{1}{12}\,\partial_X^3 U + U\,\partial_X U + \frac{1}{2T}\,U \;=\; 0,$$

as

$$U(X,T) = T^{-1}\,v\left(x = -XT^{-\frac{1}{2}}, t = T^{-\frac{1}{2}}\right) + \frac{1}{2}XT^{-1}.$$

We introduce the parameters $X \in \mathbb{R}$, $T > 0$ in the Fredholm determinant $F(X,T) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{X,T}|_{(0,+\infty)})$ taking $\mathcal{K}^{\mathrm{Ai}}_{X,T}$ the integral operator associated to the shifted and scaled Airy kernel

$$K^{\mathrm{Ai}}_{X,T}(\lambda,\mu) := T^{-\frac{1}{3}}K^{\mathrm{Ai}}(T^{-\frac{1}{3}}(\lambda+X), T^{-\frac{1}{3}}(\mu+X)).$$

$$\downarrow$$

[Flaschka - Newell, 1982] $U(X,T) = \partial_X^2 \log F(X,T) = -T^{-2/3}q(XT^{-1/3})^2$ solves cKdV.

# Outline

# Deformation of the Airy kernel

The finite temperature Airy kernel is given by

$$K^{\mathrm{Ai}}_{\sigma,s}(x,y) := \int_{-\infty}^{+\infty} \sigma(z)\mathrm{Ai}(x+s+z)\mathrm{Ai}(y+z+s)dz, \quad \sigma = \sigma_\alpha(z) = \frac{1}{1+e^{-\alpha z}}$$

and it defines a new determinantal point process.

**Remark** The Fredholm determinant $F_\sigma(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{\sigma,s}|_{(0,+\infty)})$ describes certain distributions in

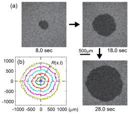
$\rightsquigarrow$ [Dean - LeDoussal - Majumdar - Schehr, 2016] free fermions at finite temperature;

$\rightsquigarrow$ [Johansson, 2008, Liechty - Wang, 2018] Moshe-Neureberg-Shapiro model.

$\rightsquigarrow$ [Amir - Corwin - Quastel, 2011] The *narrow wedge* solution of the stochastic PDE named as Kardar-Parisi-Zhang

$$\partial_T h(X,T) = \frac{1}{2}\partial_X^2 h(X,T) - \frac{1}{2}(\partial_X h(X,T))^2 + \xi(X,T)$$

where $\xi(X,T)$ is a Gaussian space-time white noise.




[Takeuchi - Sano, 2018]: turbolent liquid cristals.

# Integrability results

1. [Amir - Corwin - Quastel, 2011] Generalization of the Tracy-Widom formula as

$$\frac{d^2}{ds^2}\log F_\sigma(s) = -\int_{\mathbb{R}} \varphi^2(r;s)\sigma'(r)dr$$

   where $\varphi$ is solution of the so called integro-differential Painlevé II equation

$$\frac{\partial^2}{\partial s^2}\varphi(z;s) = \left(z + s + 2\int_{\mathbb{R}} \varphi^2(r;s)\sigma'(r)dr\right)\varphi(z;s).$$

   with $\varphi(z;s) \sim \mathrm{Ai}(z+s)$ per $s \to +\infty$ pointwise in $z$.

   **Remark** For $\sigma = \chi_{(0,+\infty)}$ then $\varphi(0;s) = q(s)$ and this goes back to the Tracy-Widom formula.

2. [Cafasso - Claeys - Ruzza, 2021] For $\sigma = \sigma_T(u) = \sigma(T^{1/3}u)$ within a certain family of functions, with given asymptotic properties, $V_\sigma(X,T) = \partial_x^2 \log\det(1 - \mathcal{K}^{\mathrm{Ai}}_{\sigma_T, -XT^{-1/3}}|_{(0,+\infty)})$ is a cKdV solution.

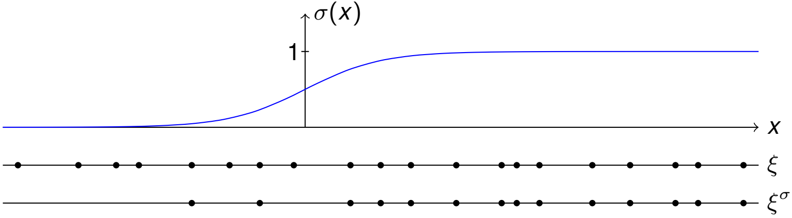
# *Thinning* operation

**Remark** Let $\sigma$ be such that $\sqrt{\sigma}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{\sigma}$ is trace-class, then

$$\downarrow$$

$$F_\sigma(s) = \det\left(1 - \mathcal{K}^{\mathrm{Ai}}_{\sigma,s}|_{(0,+\infty)}\right) = \det\left(1 - \sqrt{\sigma}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{\sigma}\right).$$

Every configuration $\xi^\sigma$ in the Airy process *$\sigma$-thinned* $\mathbb{P}^\sigma_{\mathrm{Ai}_s}$ is obtained from a random configuration $\xi$ in $\mathbb{P}_{\mathrm{Ai}_s}$, by independently eliminating a point in $\xi_j$ in the configuration $\xi$ with probability $1 - \sigma(\xi_j)$ and keeping it with probability $\sigma(\xi_j)$.




$\rightsquigarrow$ $F_\sigma(s) = \det(1 - \sqrt{\sigma}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{\sigma})$ is the gap probability of $\mathbb{P}^\sigma_{\mathrm{Ai}_s}$.
$\rightsquigarrow$ Every configuration of $\mathbb{P}^\sigma_{\mathrm{Ai}_s}$ has a. s. $\#$ points $< \infty$.

# Outline

# Janossy densities of the thinned process

For every $m \geq 1$, given $V = \{v_1, \dots, v_m\}$, the Janossy densities of the thinned process $\mathbb{P}^{\sigma}_{\mathrm{Ai}_s}$ are defined as

$$J_\sigma(V; s) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \rho_s(\lambda_1, \dots, \lambda_n, v_1, \dots, v_m) \prod_{i=1}^{m} \sigma(\lambda_i) d\lambda_i.$$

**Remark** $J_\sigma(V; s)\mathrm{d}v_1 \dots \mathrm{d}v_m$ is interpreted as the probability of having exactly $m$ points, each one in $[v_i, v_i + dv_i]$.

$J_\sigma(V; s)$ is factorized as

$$J_\sigma(V; s) = \det(1 - \sigma\mathcal{K}_s^{\mathrm{Ai}}) \det_{1 \leq k,h \leq m}(L^{\mathrm{Ai}}_{\sigma,s}(v_k, v_h)) = F_\sigma(s) \det_{1 \leq k,h \leq m}(L^{\mathrm{Ai}}_{\sigma,s}(v_k, v_h))$$

where $L^{\mathrm{Ai}}_{\sigma,s}$ is the kernel of the integral operator $\mathcal{L}^{\mathrm{Ai}}_{\sigma,s}$ defined by

$$\mathcal{L}^{\mathrm{Ai}}_{\sigma,s} := \mathcal{K}_s^{\mathrm{Ai}} \left(1 - \sigma\mathcal{K}_s^{\mathrm{Ai}}\right)^{-1}.$$

# $J_\sigma(V; s)$ and Darboux transformations

**Theorem (Claeys, Glesner, Ruzza, T.)**

$$\frac{d^2}{ds^2}\log J_\sigma(V;s) = \int_{\mathbb{R}} \varphi_\sigma(\lambda; s, V)^2 \left(-\sigma'(\lambda) + \sum_{i=1}^{m} \frac{2(1-\sigma(\lambda))}{\lambda - v_i}\right) d\lambda = u(s,V) + \frac{s}{2},$$

*where $\varphi(z; s, V)$ solves the Stark equation*

$$\left[\partial_s^2 + 2u(s,V) - s\right]\varphi(z; s, V) = z\varphi(z; s, V),$$

*with $z \to \infty$ behavior described in terms of the Airy function.*

**Remark**

- $\varphi(z; s) = \varphi(z; s, \emptyset)$ è is the solution of the integro-differential PII equation.
- $\varphi(z; s, V)$ is obtained as a Darboux transformation of $\varphi(z; s, \emptyset)$:

$$\varphi(z; s, V) = \left(1 - \sum_{i,j=1}^{m} \frac{\left(\mathbf{L}^{-1}(s,V)\right)_{j,i}}{z - v_j} \varphi(v_i; s, \emptyset)\partial_s\varphi(v_j; s, \emptyset)\right)\varphi(z; s, \emptyset)$$
$$+ \left(\sum_{i,j=1}^{m} \frac{\left(\mathbf{L}^{-1}(s,V)\right)_{j,i}}{z - v_j} \varphi(v_i; s, \emptyset)\varphi(v_j; s, \emptyset)\right)\partial_s\varphi(z; s, \emptyset).$$

# New solutions of cKdV

## Theorem (Claeys, Glesner, Ruzza, T.)

*For every* $V$, $U = U_\sigma(X,T;V) := \frac{\partial^2}{\partial X^2}\log J_\sigma(V;X,T)$ *solves cKdV*

$$\frac{\partial U}{\partial T} + \frac{1}{12}\frac{\partial^3 U}{\partial X^3} + U\frac{\partial U}{\partial X} + \frac{U}{2T} = 0.$$

**Remark** For every $T_0 > 0$, uniformly for $T \geq T_0$ and $\frac{X}{T\log^2|X|} \to -\infty$ we have

$$U_\sigma(X,T;V) = U_\sigma(X,T;\emptyset) + \frac{1}{\sqrt{|X|T}} \underbrace{\sum_{j=1}^{m} \cos\left(\frac{4|X|^{\frac{3}{2}}}{3T^{\frac{1}{2}}}(1+A_{X,T}) - \frac{2|X|^{\frac{1}{2}}}{T^{\frac{1}{2}}}v_j(1+B_{X,T}(v_j)) + O(|X|^{-1})\right)}_{\text{superposition of } m \text{ 1-solitons } U_0(X,T;v)}.$$

| DPP | (c)KdV |
|---|---|
| gap probability $F(X,T)$ | solution $U(X,T;\emptyset)$ |
| $\downarrow$ | $\downarrow$ |
| Janossy densities $J_\sigma(V;X,T)$ | Darboux transformations $U(X,T;V)$ |

Grazie!