Integrable features of sine and Airy kernels at finite temperature

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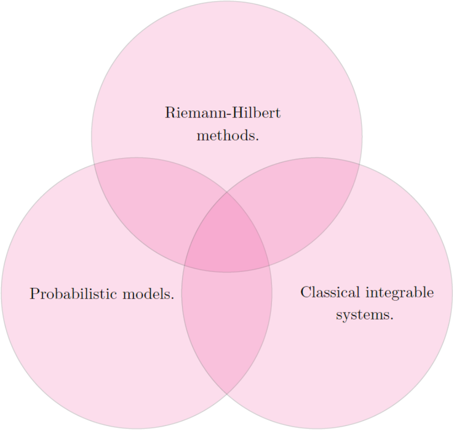
16 September 2023

Integrable probability via Riemann–Hilbert problems



Outline

G(aussian) U(nitary) E(nsamble)

GUE is the random matrix ensemble defined by taking on the set of square Hermitian matrices of size N the probability distribution

$$\mathbb{P}(M)dM = \frac{1}{Z_N} e^{-\operatorname{tr} M^2/2} dM,$$

where Z_N denotes the partition function.

Remark A square matrix of size N such that $M = \frac{X+X^\dagger}{2}$ where $X_{h,k} \sim \mathcal{N}(0,1) + i\mathcal{N}(0,1)$, are i.i.d. with

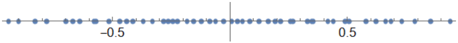
$$\mathcal{N}(0,1) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \quad \text{Gaussian distribution}$$

is distributed as in GUE.

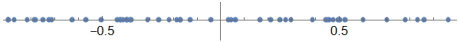
Problem Study the eigenvalues of GUE and their limiting behavior for $N \to \infty$.

$$\downarrow$$

Point processes on $\mathbb{R}$.



Eigenvalues of a GUE matrix (scaled by $2\sqrt{N}$), $N = 50$.



$N = 50$ points taken with uniform distribution in $[-1, 1]$.

Eigenvalues behavior

The eigenvalues of a GUE matrix are a determinantal point process (on $\mathbb{R}$).

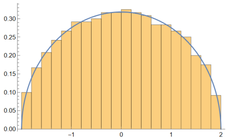
$$\downarrow$$

Meaning : for any $k \leq N$ Borel subsets $A_j \subset \mathbb{R}, j = 1, \dots, k$ pairwise disjoint, the expected number of distinct ordered eigenvalues k-tuples $(\lambda_1, \dots, \lambda_k)$ s.t. $\lambda_i \in A_i$ is given by

$$m_k(A_1 \times \cdots \times A_k) = \int_{A_1} \cdots \int_{A_k} \underbrace{\det_{1 \leq i,j \leq k} K_N(x_i, x_j)}_{= \rho_k^{(N)}(x_1, \dots, x_k) \text{ correlation function}} dx_1 \dots dx_k.$$

where the correlation kernel K_N is explicitely written in terms of the Hermite polynomials.

Remark The large N limiting behavior of the eigenvalues density of GUE matrices $\rho_1^{(N)}(x)/N$ converges to the Wigner semicircle law.



Bulk vs edge behavior large N limit

[Gaudin - Metha, Forrester 1993] For $N \to \infty$ then

- the limiting behavior of K_N in the bulk of the spectrum is

$$(Nd(x_0))^{-1} K_N \left(x_0 + \frac{x}{Nd(x_0)}, x_0 + \frac{y}{Nd(x_0)} \right) \to \underbrace{K^{\sin}(x,y)}_{\text{sine kernel}} = \int_{-1/2}^{1/2} \mathrm{e}^{2\pi \mathrm{i}(x-y)u} \mathrm{d}u,$$

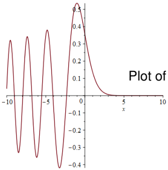
- the limiting behavior of K_N near the edge of the spectrum is

$$N^{-2/3} K_N \left(2 + \frac{x}{N^{2/3}}, 2 + \frac{y}{N^{2/3}} \right) \to \underbrace{K^{\mathrm{Ai}}(x,y)}_{\text{Airy kernel}} = \int_0^{+\infty} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\mathrm{d}u.$$

Remark $\mathrm{Ai}(x)$ is a real solution of the Airy equation

$$v''(x) = xv(x)$$

that can be written as $\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^{+\infty} \cos\left(\frac{t^3}{3} + xt \right) dt$.



Plot of $\mathrm{Ai}(x)$.

Sine and Airy determinantal point processes

For a given correlation kernel, we can associate a (self-adjoint) integral operator $\mathcal{K}$ on $L^2(\mathbb{R})$ s.t.

$$\mathcal{K}f(x) = \int_{\mathbb{R}} K(x, y)f(y)dy.$$

[Soshnikov, 2000] Hermitian locally trace class operator $\mathcal{K}$ on $L^2(\mathbb{R})$ with kernel $K(\cdot, \cdot)$ defines a determinantal point process on $\mathbb{R}$ if and only if $0 \leq \mathcal{K} \leq 1$. If the corresponding point process exists it is unique.

$$\downarrow$$

The sine and Airy kernel define two determinantal point processes. They are considered universal since they describe also limiting behavior in other models

$\rightsquigarrow$ free fermions at zero temperature;

$\rightsquigarrow$ random partitions and permutations;

$\rightsquigarrow$ percolation models.

Gap probabilities

The gap probability is the probability to find no points in a given subset of $\mathbb{R}$.

A foundamental result from the theory of DPPs says that for any Borel subset $B \subset \mathbb{R}$ (such that $\mathcal{K}|_B$ is trace-class) then the gap probability for B is given by the Fredholm determinant

$$\mathbb{P}\left(\#_B = 0\right) = 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} \int_{B^k} \det_{i,j=1}^{k} K(x_i, x_j) dx_1 \dots dx_k = \det(1 - \mathcal{K}|_B).$$

Remark Of particular interest for the sine and Airy DPPs are

$$F(s) = \det\left(1 - \mathcal{K}^{\sin}|_{(-s/2\pi, s/2\pi)}\right) \quad \text{and} \quad G(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}|_{(s,\infty)})$$

which describe respectively in the GUE setting

$$\lim_{N\to\infty} \mathbb{P}\left(\{\sqrt{N}\lambda_i^{\text{GUE}}\}_{i=1}^N \notin (-s/2\pi, s/2\pi)\right) = F(s),$$

$$\lim_{N\to\infty} \mathbb{P}\left(\left(\lambda_{max}^{\text{GUE}} - \sqrt{2N}\right)\left(\sqrt{2}N^{1/6}\right) \leq s\right) = G(s).$$

Tracy-Widom formula

The Fredholm determinant $G(s)$ satisfies

$$\frac{d^2}{ds^2}\log G(s) = -q^2(s)$$

where q is the Hastings-McLeod solution of the Painlevé II equation, i.e. it solves

$$q''(s) = sq(s) + 2q^3(s)$$

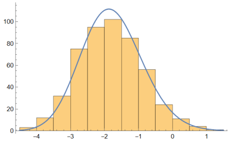
together with the boundary condition

$$q(s) \sim \mathrm{Ai}(s) \ \text{ for } \ s \to +\infty.$$

Remark In particular

$$G(s) = \exp\left(-\int_s^{+\infty}(r-s)q^2(r)dr\right)$$

and $G(s)$ is called Tracy-Widom distribution.



TW distribution and 500 samples of maximum rescaled eigenvalues of GUE matrices $N = 20$.

Jimbo-Miwa-Mori-Sato formula

The Fredholm determinant $F(s)$ satisfies

$$\frac{d}{ds} s \frac{d}{ds} \log F(s) = \nu'(s)$$

where ν is a solution of the Painlevé V σ-form equation, i.e. it solves

$$(s\nu'')^2 + 4(s\nu' - \nu)(s\nu' - \nu + (\nu')^2) = 0,$$

together with boundary condition

$$\nu(s) = -\frac{1}{\pi} s + O(s^2), \qquad s \to 0.$$

Remark In particular

$$F(s) = \exp\left(\int_0^s \frac{\nu(x)}{x} \mathrm{d}x \right).$$

Painlevé equations in a nutshell

- [Picard, 1889 - Painlevé, 1900 - Gambier, 1910] Painlevé equations are 6 nonlinear second order ODEs which do not have movable branch points and generically do not admit solutions in terms of classical special functions $\rightarrow$ their solutions are called Painlevé transcendents.
- [Jimbo - Miwa - Ueno, 1981] Every Painlevé equation admits (at least) one isomonodromic Lax pair

$$\begin{cases} \dfrac{\partial \Psi(\lambda, t)}{\partial \lambda} = A(\lambda, t)\Psi(\lambda, t) \\ \dfrac{\partial \Psi(\lambda, t)}{\partial t} = B(\lambda, t)\Psi(\lambda, t) \end{cases} \quad \text{s.t.} \quad \frac{\partial A(\lambda, t)}{\partial t} - \frac{\partial B(\lambda, t)}{\partial \lambda} + [A(\lambda, t), B(\lambda, t)] = 0$$

is equivalent to the Painlevé equation,

and it has an Hamiltonian formulation related to the so-called σ-forms.

Common structures

- They both have integral representations

$$K^{\sin}(x,y) = \int_{-1/2}^{1/2} \mathrm{e}^{2\pi i x u}\mathrm{e}^{-2\pi i y u} du \qquad K^{\mathrm{Ai}}(x,y) = \int_0^{+\infty} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u) du.$$

- Thus their integral operators can both be decomposed as

$$\mathcal{K}^{\sin} = \mathcal{F}^* \chi_{(-1/2,1/2)} \mathcal{F}$$

for $\mathcal{F}$ being the Fourier transform and $\chi_{(-1/2,1/2)}$ the projection on the interval $(-1/2, 1/2)$.

$$\mathcal{K}^{\mathrm{Ai}} = \mathcal{A} \chi_{(0,\infty)} \mathcal{A}^*$$

for $\mathcal{A}$ being the Airy transform and $\chi_{(0,\infty)}$ the projection on the interval $(0, \infty)$.

- They are both of integrable IIKS type

$$K(x,y) = \frac{\vec{f}^{\top}(x)\vec{g}(y)}{x-y}, \quad \text{with } \vec{f}^{\top}(x)\vec{g}(x) = 0.$$

In particular

$$K^{\sin}(x,y) = \frac{\mathrm{e}^{i\pi x}\mathrm{e}^{-i\pi y} - \mathrm{e}^{-i\pi x}\mathrm{e}^{i\pi y}}{2\pi i(x-y)}, \qquad K^{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

Its-Izergin-Korepin-Slavnov theory

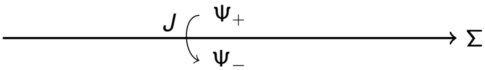
[Its - Izergin - Korepin - Slavnov, 1991] The resolvent of an integrable operator $\mathcal{K}$ is characterized through the solution of a 2×2 matrix-valued Riemann–Hilbert problem (Σ, J) where

$$J(\lambda) = I - 2\pi i \vec{f}(\lambda) \vec{g}^{\top}(\lambda), \; \lambda \in \Sigma.$$

$\rightsquigarrow$ Suppose that there is explicit dependence on a deformation parameter s, $K \to K_s = K(x, y; s)$ so that the operator $\mathcal{K}_s$ is trace-class. Then the logarithmic derivative of the associated Fredholm determinant

$$\partial_s \log \det (1 - \mathcal{K}_s) = -\mathrm{Tr}_{L^2(\Sigma)} \left((1 - \mathcal{K}_s)^{-1} \partial_s \mathcal{K}_s \right)$$

can be as well characterized by the solution of the Riemann–Hilbert.



Outline

Deformation of the kernels

$\rightsquigarrow$ For a smooth function $w : \mathbb{R} \to [0,1]$ fast decaying to zero at $-\infty$, the deformed Airy kernel

$$K^{\mathrm{Ai}}_{w,s}(x,y) = \int_{-\infty}^{+\infty} w(z)\mathrm{Ai}(x+s+z)\mathrm{Ai}(y+z+s)dz$$

and the associated Fredholm determinant $G_w(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{w,s}|_{(0,+\infty)})$.

$\rightsquigarrow$ For Schwartz function $w : \mathbb{R} \to [0,1]$, we consider the deformed sine kernel

$$K^{\sin}_{w}(x,y) = \int_{-\infty}^{\infty} w(u)\mathrm{e}^{2\pi i(x-y)u}\mathrm{d}u$$

and the associated Fredholm determinant $F_w(s) = \det\left(1 - \mathcal{K}^{\sin}_{w}|_{(-s,s)}\right)$.

Remark They both define new DPPs.

Their appearences

The deformed Airy kernel together with the analogue deformed sine kernel have been found in multiple models for specific choices of the weight function w.

- [Dean - Ledoussal - Majumdar - Schehr, 2018] Here $w(r) = (1 + e^{-\alpha r})^{-1}$ $w(r) = (1 + e^{-\lambda(4r^2-1)})^{-1}$. The limiting behavior of the positions of a system of free fermions at finite temperature trapped with certain class of potentials $(V(x) \sim x^{2n})$ in the bulk / edge (α, λ are proportional to the inverse temperature).

- [Johansson, 2008, Lietchy - Wang, 2018] Here w is essentially the same as above. The limiting behavior of the eigenvalues in the Moshe-Neurberg-Shapiro model in the bulk / edge.

- [Bothner - Little, 2022] Here $w(r) = \Phi(s\sigma^{-1}(r+1)) - \Phi(s\sigma^{-1}(r-1))$ with $\Phi(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} \mathrm{e}^{-y^2} dy$ (and σ a parameter of the model). The limiting (weak non-hermiticity limit) behavior of the real parts of eigenvalues of the Complex Elliptic Ginibre ensemble in the bulk / edge.

Integrable structures (again)

$\mathcal{K}_w^{\sin}$ and $\mathcal{K}_{w,s}^{\mathrm{Ai}}$ factorize again as their *zero temperature* versions

$$\mathcal{K}_w^{\sin} = \mathcal{F}^* w_s \mathcal{F}, \quad \text{and} \quad \mathcal{K}_w^{\mathrm{Ai}} = \mathcal{A}_s^* w \mathcal{A}_s,$$

where w_s denotes the multiplication operator by $w(\frac{\cdot}{2s})$ and $\mathcal{A}_s$ is the s-shifted Airy transform.

$$\downarrow$$

Thanks to that, composing with the projection operators and exploiting

$$\det(1 - AB) = \det(1 - BA)$$

we have

$$F_w(s) = \det\left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right), \quad \text{and} \quad G_w(s) = \det\left(1 - \sqrt{w}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{w}\right)$$

where $\sqrt{w_s}$ denotes the multiplication operator with a square root of the function $w(\frac{\cdot}{2s})$ and $\mathcal{K}_s^{\mathrm{Ai}}$ acts through the s-shifted Airy kernel $K^{\mathrm{Ai}}(x+s, y+s)$.

$\rightsquigarrow$ They can both studied by R-H method!

Outline

Generalization of the Tracy-Widom formula

[Amir - Corwin - Quastel, 2011] Generalization of the Tracy-Widom formula as

$$\frac{d^2}{ds^2} \log G_w(s) = -\int_{\mathbb{R}} \varphi^2(r; s) w'(r) dr$$

where φ is solution of the so called integro-differential Painlevé II equation

$$\frac{\partial^2}{\partial s^2} \varphi(z; s) = \left(z + s + 2 \int_{\mathbb{R}} \varphi^2(r; s) w'(r) dr \right) \varphi(z; s).$$

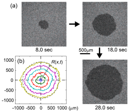
with $\varphi(z; s) \sim \mathrm{Ai}(z + s)$ per $s \to +\infty$ pointwise in z.

Remark For $w = \chi_{(0,+\infty)}$ then $\varphi(0; s) = q(s)$ and this goes back to the TW formula.

[Amir - Corwin - Quastel, 2011] The *narrow wedge* solution of the stochastic PDE named as Kardar-Parisi-Zhang

$$\partial_T h(X, T) = \frac{1}{2} \partial_X^2 h(X, T) - \frac{1}{2} (\partial_X h(X, T))^2 + \xi(X, T)$$

where $\xi(X, T)$ is a Gaussian space-time white noise.



[Takeuchi - Sano, 2018]: turbolent liquid cristals.

Connection with KdV

[Cafasso - Claeys - Ruzza, 2021] For every w within the class, the function $u(x,t) = \partial_x^2 \log G_w(x,t) + \frac{x}{2t}$ for $G_w(x,t)$ the deformed Fredholm determinant associated to

$$t^{2/3} K^{\mathrm{Ai}}_{w_t, xt^{-1}}(t^{2/3}\cdot, t^{2/3}\cdot),$$

with $w_t(\lambda) = w(t^{2/3}\lambda)$, solves the Korteweg-de Vries equation

$$\partial_t u + 2u\partial_x u + \frac{1}{6}\partial_x^3 u = 0.$$

Remark For $w = \chi_{(0,+\infty)}$, the Tracy-Widom formula gives

$$u(x,t) = -t^{-2/3} q^2(-xt^{-1/3}) + \frac{x}{2t}$$

which is an instance of the self-similarity relation between the Painlevé II equation and the KdV equation.

Generalization of the Jimbo-Miwa-Mori-Sato formula

Theorem (Claeys - T., 2023)

For every $s > 0$, we have the identity

$$\partial_s s \partial_s \log F_w(s) = \frac{1}{\pi} \int_{\mathbb{R}} \lambda w'(\lambda) \phi(\lambda; s) \psi(\lambda; s) \mathrm{d}\lambda,$$

where ϕ, ψ solve the (Zakharov-Shabat) system of equations

$$\partial_s \phi(\lambda; s) = \mathrm{i}\lambda \phi(\lambda, s) - \frac{1}{2\pi \mathrm{i} s} \int_{\mathbb{R}} \phi^2(\mu; s) w'(\mu) \mathrm{d}\mu \, \psi(\lambda; s),$$

$$\partial_s \psi(\lambda; s) = \frac{1}{2\pi \mathrm{i} s} \int_{\mathbb{R}} \psi^2(\mu; s) w'(\mu) \mathrm{d}\mu \, \phi(\lambda; s) - \mathrm{i}\lambda \psi(\lambda; s).$$

with $\lambda \to \pm\infty$ asymptotics $\phi(\lambda; s) \sim \mathrm{e}^{\mathrm{i}s\lambda}, \psi(\lambda; s) \sim \mathrm{e}^{-\mathrm{i}s\lambda}$.

Remark If w is even, then

$$\partial_s s \partial_s \log F_w(s) = \frac{1}{\pi} \int_{\mathbb{R}} \lambda w'(\lambda) \phi(\lambda; s) \phi(-\lambda; s) \mathrm{d}\lambda,$$

where ϕ solves the integro-differential equation

$$\partial_s \phi(\lambda; s) = \mathrm{i}\lambda \phi(\lambda; s) - \frac{1}{2\pi \mathrm{i} s} \int_{\mathbb{R}} \phi^2(\mu; s) w'(\mu) \mathrm{d}\mu \, \phi(-\lambda; s).$$

Reduction to Painlevé V

Back to the case $w(r) = \chi_{(-\frac{1}{2},\frac{1}{2})}$, $w'(r) = \delta_{-\frac{1}{2}}(r) - \delta_{\frac{1}{2}}(r)$.

Then our integro-differential equations reduce to

$$\partial_s \phi\left(\pm\frac{1}{2};s\right) = \pm\frac{\mathrm{i}}{2}\phi\left(\pm\frac{1}{2};s\right) - \frac{1}{2\pi\mathrm{i}s}\left(\phi^2\left(-\frac{1}{2};s\right) - \phi^2\left(\frac{1}{2};s\right)\right)\phi\left(\mp\frac{1}{2};s\right)$$

and by defining

$$v(x) = \frac{1}{2\pi\mathrm{i}}\phi\left(\frac{1}{2};\frac{x}{2\mathrm{i}}\right)\phi\left(-\frac{1}{2};\frac{x}{2\mathrm{i}}\right), \qquad u(x) = \frac{\phi^2\left(\frac{1}{2};\frac{x}{2\mathrm{i}}\right)}{\phi^2\left(-\frac{1}{2};\frac{x}{2\mathrm{i}}\right)},$$

we recover the system

$$xv' = v^2(u - \frac{1}{u}), \quad xu' = xu - 2v(u-1)^2$$

implying that u solves the Painlevé V equation

$$u'' = \frac{u}{x} - \frac{u'}{x} - \frac{u(u+1)}{2(u-1)} + (u')^2\frac{3u-1}{2u(u-1)}.$$

Moreover, the JMMS formula is recovered by $\nu'(s) = -\frac{1}{\pi}\phi\left(\frac{1}{2};s\right)\phi\left(-\frac{1}{2};s\right)$.

Connection with a *new* PDE

Let $W : \mathbb{R} \to \mathbb{C}$ be smooth and decaying fast at $+\infty$, such that the even function $w^{(y)}(\lambda) = W(\lambda^2 - y)$ is a Schwartz function.

With a change of variable we have

$$F_{w^{(y)}}(s) = \det\left(1 - \mathcal{K}^{\sin}_{w^{(y)}}|_{[-\frac{s}{2\pi}, \frac{s}{2\pi}]}\right) = \det\left(1 - \mathcal{K}^{\sin}_{w_{y,s}}|_{[-1/2,1/2]}\right) = Q_W(y,s)$$

with $w_{y,s}(\zeta) = W\left(\frac{\pi^2\zeta^2}{s^2} - y\right)$.

Theorem (Claeys - T., 2023)

The function $q = q_W = -\partial_s^2 \log F_w(y,s)$ *solves the PDE*

$$\partial_s\left(\frac{\partial_s\partial_y q}{2q}\right) = \partial_y(q^2) - 1,$$

and $\sigma = \sigma_W = \log F_w(y,s)$ *solves the PDE*

$$(\partial_s^2\partial_y\sigma)^2 = 4\partial_s^2\sigma\left(-2s\partial_s\partial_y\sigma + 2\partial_y\sigma - (\partial_s\partial_y\sigma)^2\right).$$

Remark For the specific choice $W(r) = \frac{1}{e^{4r}+1}$, this result was found in the original paper of [Its - Izergin - Korepin - Slavnov, 1991].

Initial boundary value problem for σ

Theorem (Claeys, T. 2023)

Let $f : \mathbb{R} \to \mathbb{C}$ be C^∞ and decaying fast at $-\infty$, such that $f(y - .^2)$ is a Schwartz function for all $y \in \mathbb{R}$. Then the initial value problem for the σ-PDE with initial data

$$\lim_{s \to 0} \frac{1}{s} \sigma_W(y, s) = f(y)$$

is solved by $\sigma_W(y, s) = \log Q_W(y, s)$, with

$$W(r) = -2 \int_0^\infty f'(-u^2 - r) \mathrm{d}u.$$

Remark Notice that the W, which can be interpreted as *scattering data*, and the initial data f are related by a very simple integral transformation.

About the method

- Construct the IIKS 2×2 matrix-valued Riemann–Hilbert problem ($\Sigma = \mathbb{R}, J = J^{\mathrm{Ai,sin}}$) exploiting the structure of the kernels built respectively with vectors

$$\vec{f}_w^{\,\mathrm{Ai}}(\lambda; x, t) = \sqrt{w(\lambda)}\, \vec{f}_{x,t}^{\,\mathrm{Ai}}(\lambda), \qquad \vec{f}_w^{\,\sin}(\lambda; s, y) = \sqrt{w(\lambda)}\vec{f}_{y,s}^{\,\sin}(\lambda),$$
$$\vec{g}_w^{\,\mathrm{Ai}}(\lambda; x, t) = \sqrt{w(\lambda)}\, \vec{g}_{x,t}^{\,\mathrm{Ai}}(\lambda). \qquad \vec{g}_w^{\,\sin}(\lambda; s, y) = \sqrt{w(\lambda)}\vec{g}_{y,s}^{\,\sin}(\lambda).$$

- Construct from its solution $\Theta = \Theta(\lambda; x, t)$ and $\Psi = \Psi(\lambda; s, y)$, which give respectively the Lax pairs

$$\partial_x \Theta = B(\lambda; x, t)\Theta \qquad \partial_s \Psi = M(\lambda; s, y)\Psi$$
$$\partial_t \Theta = C(\lambda; x, t)\Theta \qquad \underbrace{D_{\lambda,y}}_{=\partial_\lambda + 2\lambda\partial_y} \Psi = L(\lambda; s, y)\Psi$$

s.t. $\partial_t B - \partial_x C + [B, C] = 0 \;\leftrightarrow\;$ KdV $\qquad$ s.t. $\partial_s L - D_{\lambda,y} M + [L, M] = 0 \;\leftrightarrow\; q -$ PDE.

Final remarks

- Other kernels of integrable type can be treated at the same way $\rightsquigarrow$ finite temperature Bessel in progress.
- The structure of Hankel composition operator of these kernels also can be exploited to use Riemann–Hilbert problems to study their Fredholm determinants and connect them to integrable equations $\rightsquigarrow$ work of [Bothner, 2022].
- Other than gap probabilities (alias Fredholm determinants) we can also study other quantities related to the DPPs Janossy densities : they corresponds to Darboux transformations of the Riemann–Hilbert problems / integrable systems behind $\rightsquigarrow$ for the Airy case [Claeys - Glesner - Ruzza - T., 2023].

Thank you!