Determinantal point processes via Riemann–Hilbert problems

Sofia Tarricone

Institut de Physique Théorique, CEA Paris-Saclay

SPA23

CS3 - Various aspects of determinantal point processes

University of Lisbon, 24 July 2023

Outline

DPP on the real line

$\rightsquigarrow$ $X \subset \mathbb{R}$ is locally finite if for every $B \subset \mathbb{R}$ bounded then $\#\{X \cap B\} < \infty$. Then a point process on $\mathbb{R}$ is a probability measure on the space of all locally finite configurations of $\mathbb{R}$.
$\rightsquigarrow$ In other words, studying the point process means studying the counting functions, random variables defined for any Borel subset $B \subset \mathbb{R}$ as

$$\#_B : \mathrm{Conf}(\mathbb{R}) \to \mathbb{N}, \quad \#_B(X) = \#\{X \cap B\}.$$

$\rightsquigarrow$ A point process admits correlation functions $\rho_k, k \geq 1$ if the multiplicative statistics of the counting functions of any k pairwise disjoint Borel subset $A_j, j = 1, \ldots, k$ can be computed by

$$\mathbb{E}\left(\prod_{i=1}^{k} \#_{A_i}\right) = \int_{A_1} \cdots \int_{A_k} \rho_k(x_1, \ldots, x_k) dx_1 \ldots dx_k.$$

Remark [Lenard, 1973 – 75] studied the problem of defining a point process via its correlation functions.

$\rightsquigarrow$ A determinantal point process is a point process with correlation functions given by

$$\rho_k(x_1, \ldots, x_k) = \det_{i,j=1}^{k} \left(K(x_i, x_j)\right)$$

for $K(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{C}$ an Hermitian kernel.

The Airy and sine kernel

For a given correlation kernel, we can associate a (self-adjoint) integral operator $\mathcal{K}$ on $L^2(\mathbb{R})$ s.t.

$$\mathcal{K}f(x) = \int_{\mathbb{R}} K(x,y)f(y)dy.$$

[Soshnikov, 2000] Hermitian locally trace class operator $\mathcal{K}$ on $L^2(\mathbb{R})$ with kernel $K(\cdot,\cdot)$ defines a determinantal point process on $\mathbb{R}$ if and only if $0 \leq \mathcal{K} \leq 1$. If the corresponding point process exists it is unique.

$$\downarrow$$

The sine and Airy kernels are functions of two variables $(x, y) \in \mathbb{R}^2$ defined respectively as

$$K^{\sin}(x,y) = \frac{\mathrm{e}^{\pi \mathrm{i} x}\mathrm{e}^{-\pi \mathrm{i} y} - \mathrm{e}^{-\pi \mathrm{i} x}\mathrm{e}^{\pi \mathrm{i} y}}{2\pi \mathrm{i}(x-y)} = \int_{-1/2}^{1/2} \mathrm{e}^{2\pi \mathrm{i}(x-y)u}\mathrm{d}u \quad \left(= \frac{\sin(\pi(x-y))}{\pi(x-y)}\right),$$

$$K^{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y} = \int_0^{+\infty} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\mathrm{d}u.$$

They both define determinantal point processes on the real line with almost surely an infinite number of particles.

Gap probabilities

By gap probability we mean the probability to find no points in a given subset of $\mathbb{R}$.

A foundamental result from the theory of DPPs says that for any Borel subset $B \subset \mathbb{R}$ (such that $\mathcal{K}|_B$ is trace-class) then the gap probability for B is given by the Fredholm determinant

$$\mathbb{P}\left(\#_B = 0\right) = 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} \int_{B^k} \det_{i,j=1}^{k} K(x_i, x_j) dx_1 \dots dx_k = \det(1 - \mathcal{K}|_B).$$

Remark Of particular interest for the sine and Airy DPPs are the following gap probabilities

$$F(s) = \det\left(1 - \mathcal{K}^{\sin}|_{(-s,s)}\right) \quad \text{and} \quad G(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}|_{(s,\infty)}).$$

GUE: bulk vs edge behavior

The Gaussian Unitary Ensemble is built up by considering the set of Hermitian $N \times N$ matrices, together with

$$\mathbb{P}(M)dM = \frac{1}{Z_N} e^{-\operatorname{tr} M^2/2} dM,$$

$$\downarrow$$

The eigenvalues of GUE matrices are described by a determinantal point process where the correlation kernel K_N is written in terms of Hermite polynomials.

[Gaudin - Metha, Forrester 1993] In the large N limit, the behavior of the eigenvalues' correlation kernels is

Bulk

$$(Nd(x_0))^{-1} K_N \left(x_0 + \frac{x}{Nd(x_0)}, x_0 + \frac{y}{Nd(x_0)} \right)$$

$$\downarrow \; N \to \infty$$

$$K^{\sin}(x, y)$$

Edge

$$N^{-2/3} K_N \left(2 + \frac{x}{N^{2/3}}, 2 + \frac{y}{N^{2/3}} \right)$$

$$\downarrow \; N \to \infty$$

$$K^{\mathrm{Ai}}(x, y)$$

In particular

$$\lim_{N\to\infty} \mathbb{P}\left(\{\sqrt{N}\lambda_i^{\mathrm{GUE}}\}_{i=1}^N \notin (-s, s) \right) = F(s), \quad \lim_{N\to\infty} \mathbb{P}\left(\left(\lambda_{max}^{\mathrm{GUE}} - \sqrt{2N}\right) \left(\sqrt{2}N^{1/6}\right) \leq s \right) = G(s).$$

The deformed Airy and sine kernel

$\rightsquigarrow$ For a smooth function $w : \mathbb{R} \to [0, 1]$ fast decaying to zero at $-\infty$, the deformed Airy kernel

$$K^{\mathrm{Ai}}_{w,s}(x, y) = \int_{-\infty}^{+\infty} w(z)\mathrm{Ai}(x + s + z)\mathrm{Ai}(y + z + s)dz$$

and the associated Fredholm determinant $G_w(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{w,s}|_{(0,+\infty)})$.

$\rightsquigarrow$ For a smooth, integrable function $w : \mathbb{R} \to [0, 1]$, we consider the deformed sine kernel

$$K^{\sin}_{w}(x, y) = \int_{-\infty}^{\infty} w(u)\mathrm{e}^{2\pi i(x-y)u}\mathrm{d}u$$

and the associated Fredholm determinant $F_w(s) = \det\left(1 - \mathcal{K}^{\sin}_{w}|_{(-s,s)}\right)$.

Remark They both define new DPPs.

And their appearences

The deformed Airy kernel together with the analogue deformed sine kernel have been found in multiple models for specific choices of the weight function w.

- [Dean - Ledoussal - Majumdar - Schehr, 2018] Here $w(r) = (1 + e^{-\alpha r})^{-1}$ $w(r) = (1 + e^{-\lambda(4r^2-1)})^{-1}$. The limiting behavior of the positions of a system of free fermions at finite temperature trapped with certain class of potentials ($V(x) \sim x^{2n}$) in the bulk / edge (α, λ are proportional to the inverse temperature).

- [Johansson, 2008, Lietchy - Wang, 2018] Here w is essentially the same as above. The limiting behavior of the eigenvalues in the Moshe-Neurberg-Shapiro model in the bulk / edge.

- [Bothner - Little, 2022] Here $w(r) = \Phi(s\sigma^{-1}(r+1)) - \Phi(s\sigma^{-1}(r-1))$ with $\Phi(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} \mathrm{e}^{-y^2} dy$. The limiting (weak non-hermiticity limit) behavior of the real parts of eigenvalues of the Complex Elliptic Ginibre ensemble in the bulk / edge.

Aim : studying the functions $F_w(s), G_w(s)$.

Outline

Its-Izergin-Korepin-Slavnov theory

An integral operator $\mathcal{K}$ acting on $L^2(\Sigma), \Sigma \subset \mathbb{R}$ is said of integrable IIKS form when its kernel can be written in the form

$$K(x,y) = \frac{\vec{f}^\top(x)\vec{g}(y)}{x-y}, \quad \text{with} \quad f^\top(x)g(x) = 0,$$

for some (k)vector-valued functions $\vec{f}(x), \vec{g}(x)$.

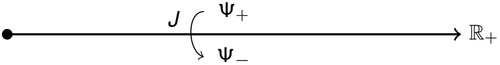
[Its - Izergin - Korepin - Slavnov, 1991] The resolvent of such an operator $\mathcal{K}$ is characterized through the solution of a $k \times k$ matrix-valued Riemann–Hilbert problem (Σ, J) where

$$J(x) = I - 2\pi i f(x) g^\top(x).$$

Moreover, suppose there is explicit dependence on a parameter s, $K \to K_s = K(x,y;s)$ and the operator $\mathcal{K}_s$ is trace-class. Then the logarithmic derivative of the associated Fredholm determinant

$$\partial_s \log \det (1 - \mathcal{K}_s) = -\mathrm{Tr}_{L^2(\Sigma)} \left((1 - \mathcal{K}_s)^{-1} \partial_s \mathcal{K}_s \right)$$

can be as well characterized by the solution of the Riemann–Hilbert.



Tracy-Widom formulas for the gap probabilities

Remark Both $\mathcal{K}^{\sin}$ and $\mathcal{K}^{\mathrm{Ai}}$ are integrable IIKS operators and thus the Fredholm determinants $F(s)$ and $G(s)$ can be studied via the Riemann–Hilbert method.

[Jimbo-Miwa-Mori-Sato, 1980]

The Fredholm determinant $F(s)$ satisfies

$$F(s) = \exp\left(\int_0^{\pi s} \frac{\nu(x)}{x} \mathrm{d}x\right),$$

where ν is a solution of the Painlevé V σ-form equation, i.e. it solves

$$(s\nu'')^2 + 4(s\nu' - \nu)(s\nu' - \nu + (\nu')^2) = 0,$$

together with boundary condition

$$\nu(s) = -\frac{1}{\pi} s + O(s^2), \qquad s \to 0.$$

[Tracy-Widom, 1994]

The Fredholm determinant $G(s)$ satisfies

$$G(s) = \exp\left(-\int_s^{+\infty} (r-s)u^2(r)dr\right)$$

where u is the Hastings-McLeod solution of the Painlevé II equation, i.e. it solves

$$u''(s) = su(s) + 2u^3(s)$$

together with the boundary condition

$$u(s) \sim \mathrm{Ai}(s) \text{ for } s \to +\infty.$$

Factorization property

Both $\mathcal{K}^{\mathrm{Ai}}, \mathcal{K}^{\sin}$ can be factorized in a useful way. Respectively

$$\mathcal{K}^{\sin} = \mathcal{F}^* \chi_{(-1/2,1/2)} \mathcal{F} \text{ and } \mathcal{K}^{\mathrm{Ai}} = \mathcal{A} \chi_{(0,\infty)} \mathcal{A}^*$$

for $\mathcal{F}$ being the Fourier transform, $\chi_{(-1/2,1/2)}$ the projection on the interval $(-1/2, 1/2)$ and $\mathcal{A}$ being the Airy transform and $\chi_{(0,\infty)}$ the projection on the interval $(0, \infty)$.
Similarly for $\mathcal{K}_w^{\sin}|_{(-s,s)}$ and $\mathcal{K}_{w,s}^{\mathrm{Ai}}|_{(0,\infty)}$ replacing the projections by multiplications by w and composing with the appropriate projection.

Remark Thanks to that we have

$$F_w(s) = \det\left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right), \text{ and } G_w(s) = \det\left(1 - \sqrt{w}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{w}\right)$$

where $\sqrt{w_s}$ denotes the multiplication operator with a square root of the function $w(\frac{\cdot}{2s})$ and $\mathcal{K}_s^{\mathrm{Ai}}$ acts through the s-shifted Airy kernel $K^{\mathrm{Ai}}(x+s, y+s)$. $\rightsquigarrow$ They can both studied by R-H method!

Generalization of Tracy-Widom formulas for the deformed versions

[Claeys - T., 2023+]

$$\partial_s s \partial_s \log F_w(s) = 2 \int_{\mathbb{R}} \lambda w'(\lambda)\phi(\lambda, s)\phi(-\lambda, s)\mathrm{d}\lambda,$$

where ϕ solves

$$\partial_s \phi(\lambda, s) = 2i\pi\lambda\phi(\lambda, s) + \frac{\mathrm{i}}{2\pi s}\int_{\mathbb{R}} \phi^2(\mu, s) w'(\mu)\mathrm{d}\mu\phi(-\lambda, s),$$

with $\lambda \to \infty$ asymptotics $\phi(\lambda, s) \sim \mathrm{e}^{2\pi i s\lambda}$.

$\rightsquigarrow$ [Bothner - Little, 2022] found an alternative formulation.

[Amir - Corwin - Quastel, 2011]

$$\partial_s^2 \log G_w(s) = -\int_{\mathbb{R}} \varphi^2(\lambda, s) w'(\lambda)\mathrm{d}\lambda,$$

where φ solves

$$\partial_s^2 \varphi(\lambda, s) = (\lambda + s)\,\varphi(\lambda, s) + 2\int_{\mathbb{R}} \varphi^2(\lambda, s) w'(\lambda)\mathrm{d}\lambda \varphi(\lambda, s).$$

with $\varphi(\lambda, s) \sim \mathrm{Ai}(\lambda + s)$ for $s \to +\infty$ pointwise in λ.

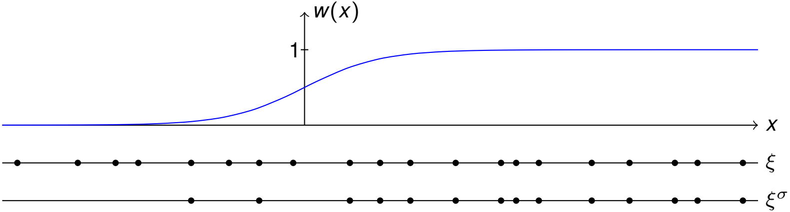
$\rightsquigarrow$ [Cafasso - Claeys - Ruzza, 2021] rederived the formula by R-H method.

Outline

Thinning the Airy DPP

The thinned shifted Airy determinantal point process $\mathbb{P}^{w}_{\mathrm{Ai}_s}$ is constructed as follows.

For every random configuration ξ in $\mathbb{P}_{\mathrm{Ai}_s}$, a configuration ξ^w in $\mathbb{P}^{w}_{\mathrm{Ai}_s}$ is built by independently eliminating a particle ξ_j in the configuration ξ with probability $1 - w(\xi_j)$ and by keeping it with probability $w(\xi_j)$:



Remark We choose w so that $\sqrt{w}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{w}$ is trace-class.
$\rightsquigarrow$ The probability of having no particles in the thinned process is given by $\det(1 - \sqrt{w}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{w}) = G_w(s)$.
$\rightsquigarrow$ Moreover, the thinned process $\mathbb{P}^{w}_{\mathrm{Ai}_s}$ has a. s. $\#$ particles $< \infty$.

Jànossy densities of the thinned Airy DPP

For a given set $V = \{v_1, \dots, v_m\}$, we can then define Janossy densities of the thinned shifted Airy point process

$$J_w(V; s) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \rho_s(\lambda_1, \dots, \lambda_n, v_1, \dots, v_m) \prod_{i=1}^{m} w(\lambda_i) d\lambda_i.$$

One can then show that $J_w(V; s)$ factorizes as

$$J_w(V; s) = \det(1 - \sqrt{w}\mathcal{K}_s^{\mathrm{Ai}}\sqrt{w}) \det_{1 \le k,h \le m}(L_{w,s}^{\mathrm{Ai}}(v_k, v_h)) = F_w(s) \det_{1 \le k,h \le m}(L_{w,s}^{\mathrm{Ai}}(v_k, v_h))$$

where $L_{w,s}^{\mathrm{Ai}}$ is the kernel of the operator $\mathcal{L}_{w,s}^{\mathrm{Ai}}$ defined as

$$\mathcal{L}_{w,s}^{\mathrm{Ai}} := \mathcal{K}_s^{\mathrm{Ai}} \left(1 - w\mathcal{K}_s^{\mathrm{Ai}}\right)^{-1}.$$

Remark We notice that this kernel is explicitely written

$$L_{w,s}^{\mathrm{Ai}}(\lambda, \mu) = \int_s^{+\infty} \varphi(\lambda, s')\varphi(\mu, s')\, \mathrm{d}s' = \frac{\varphi(\lambda, s)\varphi'(\mu, s) - \varphi'(\lambda, s)\varphi(\mu, s)}{\lambda - \mu},$$

where φ denotes the previous solution of the integro-differential Painlevé II equation.

Darboux transformations of the integro-differential Painlevé II equation

[Claeys - Glesner - Ruzza - T., 2023] We have

$$\partial_s^2 \log J_w(V;s) = \int_{\mathbb{R}} \varphi(\lambda, s; V)^2 \left(-w'(\lambda) + \sum_{i=1}^{m} \frac{2(1 - w(\lambda))}{\lambda - v_i} \right) d\lambda = u(s, V) + \frac{s}{2}.$$

Here $\varphi(\lambda, s; V)$ solves the Stark equation with potential $u(s, V)$ given above

$$\left[\partial_s^2 + 2u(s, V) - s \right] \varphi(\lambda, s; V) = \lambda \varphi(\lambda, s; V),$$

with asymptotic behavior for $\lambda \to \infty$ in terms of the Airy function.

Remark

- We have that $\varphi(\lambda, s) = \varphi(\lambda, s; \emptyset)$ the solution of the integro-differential PII.
- Moreover, $\varphi(\lambda, s; V)$ is obtained by an explicit Darboux transformation of $\varphi(\lambda, s; \emptyset)$:

$$\varphi(\lambda, s; V) = \left(1 - \sum_{i,j=1}^{m} \frac{\left(\mathbf{L}^{-1}(s, V)\right)_{j,i}}{z - v_j} \varphi(v_i, s; \emptyset) \partial_s \varphi(v_j, s; \emptyset) \right) \varphi(\lambda, s; \emptyset)$$
$$+ \left(\sum_{i,j=1}^{m} \frac{\left(\mathbf{L}^{-1}(s, V)\right)_{j,i}}{z - v_j} \varphi(v_i, s; \emptyset) \varphi(v_j, s; \emptyset) \right) \partial_s \varphi(\lambda, s; \emptyset).$$

Thank you!