The Tracy-Widom distribution for GUE and its generalizations

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Outline

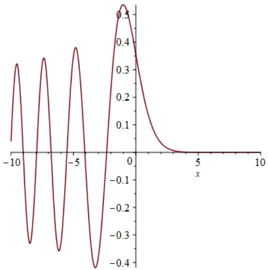
The Airy kernel

The Airy function $\mathrm{Ai}(x)$ is a rapidly decaying at $+\infty$ real solution of the Airy equation

$$v''(x) = xv(x)$$

which can be represented by

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{t^3}{3} + xt\right) dt, \;\; x \in \mathbb{R}.$$



The Airy kernel $K^{\mathrm{Ai}}(x, y)$ is then built in two equivalent ways

$$K^{\mathrm{Ai}}(x, y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y} = \int_0^{+\infty} \mathrm{Ai}(x + t)\mathrm{Ai}(y + t)dt, \;\; (x, y) \in \mathbb{R}^2$$

and the integral operator $\mathcal{K}^{\mathrm{Ai}}$ acting on $f \in L^2(\mathbb{R})$ through the Airy kernel acts like

$$\mathcal{K}^{\mathrm{Ai}} f(x) = \int_{\mathbb{R}} K^{\mathrm{Ai}}(x, y)f(y)dy.$$

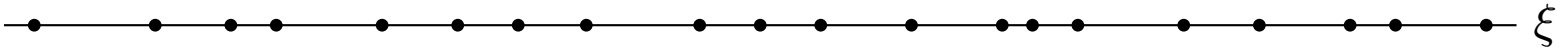
The Airy determinantal point process

[Soshnikov, 2000] Hermitian locally trace class operator $\mathcal{K}$ on $L^2(\mathbb{R})$ defines a determinantal point process on $\mathbb{R}$ if and only if $0 \leq \mathcal{K} \leq 1$. If the corresponding point process exists it is unique.

$$\downarrow$$

The Airy DPP $\mathbb{P}^{\mathrm{Ai}}$ is the point process on $\mathbb{R}$ described by correlation functions

$$\rho(x_1, \dots, x_k) = \det_{i,j=1,\dots k} K^{\mathrm{Ai}}(x_i, x_j).$$



Properties Each configuration in $\mathbb{P}^{\mathrm{Ai}}$ counts almost surely an infinite number of points and a largest point. The probability distribution particle of the largest point is given by

$$F(s) := \det\left(1 - \mathcal{K}^{\mathrm{Ai}}|_{(s,+\infty)}\right) = 1 + \sum_{n\geq 1}^{\infty} \frac{(-1)^n}{n!} \int_{(s,\infty)^n} \det_{i,j=1,\dots n} K^{\mathrm{Ai}}(x_i, x_j) dx_1 \dots dx_n.$$

GUE at the edge

The Gaussian Unitary Ensemble is built up by taking the set of Hermitian $N \times N$ matrices, together with

$$\mathbb{P}(U)dU = \frac{1}{Z_N} e^{-\operatorname{tr} U^2/2} dU.$$

$$\downarrow$$

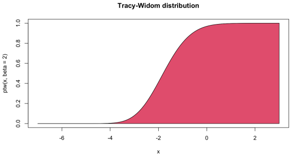
The eigenvalues of GUE matrices are described by a determinantal point process where the correlation kernel K_N is written in terms of Hermite polynomials.

[Forrester, 1993] In the large N limit, the behavior of the eigenvalues' correlation kernels near the edge of the specrtum is

$$N^{-2/3} K_N \left(2 + \frac{x}{N^{2/3}}, 2 + \frac{y}{N^{2/3}}\right) \to_{N\to\infty} K^{\mathrm{Ai}}(x, y)$$

And the limiting behavior of the distribution of the largest eigenvalue of a GUE matrix near the edge is described by

$$\lim_{N\to\infty} \mathbb{P}\left(\left(\lambda_{max}^{GUE} - \sqrt{2N}\right)\left(\sqrt{2}N^{1/6}\right) \leq s\right) = F(s).$$



The Tracy-Widom formula

[Tracy - Widom, 1994] The Fredholm determinant $F(s)$ satisfies

$$\frac{d^2}{ds^2} \ln F(s) = -u^2(s)$$

where u is the Hastings-McLeod solution of the Painlevé II equation, i.e. the unique solution of the boundary value problem

$$u''(s) = su(s) + 2u^3(s)$$

together with the condition $u(s) \sim \mathrm{Ai}(s)$ for $s \to +\infty$.

Integrating, we have

$$F(s) = \exp\left(-\int_s^{+\infty} (r-s)u^2(r)dr \right).$$

Remark [Picard, 1889 - Painlevé, 1900 - Gambier, 1910] Painlevé equations are 6 nonlinear second order ODEs which do not have movable branch points and generically do not admit solutions in terms of classical special functions.

Other occurences of $F(s)$

- [Eisler (and others), 2013] Limiting behavior of the probability distribution function of positions of free fermions trapped at zero temperature.
- [Johansson, 2003] Limiting behavior of the last passage time in directed last passage percolation models with independent identically distributed geometric weights.
- [Baik - Deift - Johansson, 1999] Limiting behavior of the first part of a random partition taken with respect to the Poissonized Plancherel measure and solution of the Ulam problem.

The Ulam problem

Consider now the symmetric group S_M taken with uniform distribution so that for any $\pi_M \in S_M$ we have

$$\mathbb{P}(\pi_M) = \frac{1}{M!}$$

and denote $\ell(\pi_M)$ the length of the longest increasing sub-sequence of π_N.

Example $\pi_5 = 4 \quad 3 \quad 1 \quad 2 \quad 5$ and $\ell(\pi_5) = 3$.

Ulam problem (1961)

Describe the behavior of $\ell(\pi_M)$ for $M \to \infty$.

[Baik - Deift - Johansson, 1999] The limiting behavior of the lenght of the longest increasing subsequence of a random permutation is

$$\lim_{M\to\infty} \mathbb{P}\left(\frac{\ell(\pi_M) - 2\sqrt{M}}{M^{1/6}} \leq s \right) = F(s).$$

Remark The connection with DPP is explained in
[Borodin - Okunkov - Olshanski , 1999].

From permutations to partitions

[Robinson–Schensted correspondence] The following map

$$RS : \pi_M \ni S_M \to RS(\pi_M) \in \{(P, Q) \in \mathrm{SYT}_M \times \mathrm{SYT}_M,\ \mathrm{sh}(P) = \mathrm{sh}(Q)\}$$

is a bijection.

Example Consider the permutation $\pi_5 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{matrix}$.

The RS correspondence associate the pair $RS(\pi_5) = (P_5, Q_5)$

$$\emptyset, \emptyset \to \begin{matrix}4\end{matrix}, \begin{matrix}1\end{matrix} \to \begin{matrix}3\\4\end{matrix}, \begin{matrix}1\\2\end{matrix} \to \begin{matrix}1\\3\\4\end{matrix}, \begin{matrix}1\\2\\3\end{matrix} \to \begin{matrix}1&2\\3&\\4&\end{matrix}, \begin{matrix}1&4\\2&\\3&\end{matrix} \to \begin{matrix}1&2&5\\3&&\\4&&\end{matrix}, \begin{matrix}1&4&5\\2&&\\3&&\end{matrix} = (P_5, Q_5).$$

$$\begin{matrix}1&2&5\\3&&\\4&&\end{matrix}, \begin{matrix}1&4&5\\2&&\\3&&\end{matrix} \to \begin{matrix}1&2\\3&\\4&\end{matrix}, \begin{matrix}1&4\\2&\\3&\end{matrix} \to \begin{matrix}1\\3\\4\end{matrix}, \begin{matrix}1\\2\\3\end{matrix} \to \begin{matrix}3\\4\end{matrix}, \begin{matrix}1\\2\end{matrix} \to \begin{matrix}4\end{matrix}, \begin{matrix}1\end{matrix} \to \emptyset, \emptyset.$$

$$x_5 = 5 \qquad x_4 = 2 \qquad x_3 = 1 \qquad x_2 = 3 \qquad x_1 = 4 \implies SR(P_5 Q_5) = \pi_5.$$

The Schensted theorem

In particular, if $\lambda(\pi_M) = (\lambda_1(\pi_M) \geq \lambda_2(\pi_M) \dots)$ is the partition coinciding with $\mathrm{sh}(P) = \mathrm{sh}(Q)$ in the above correspondence, the Schensted theorem (1961) says

$$\ell(\pi_M) = \lambda_1(\pi_M).$$

Example For π_5 we had $\lambda(\pi_5) = (3 \geq 1 \geq 1)$.

Moreover, the uniform distribution on S_M pushes forward through *RS* onto the set $\mathcal{Y}_M$ of all partitions of M inducing on it the *Plancherel* measure

$$\mathbb{P}_{\mathrm{Pl.}}(\lambda) = \frac{F_\lambda^2}{M!}, \text{ with } F_\lambda = \#\{P \in \mathrm{SYT}_M,\ \mathrm{sh}(P) = \lambda\}.$$

In this sense

$$\mathbb{P}(\ell(\pi_M) \leq k) = \mathbb{P}_{\mathrm{Pl.}}(\lambda_1(\pi_M) \leq k).$$

The Baik-Deift-Johannson result

On the set of all partitions of any size $\mathcal{Y} = \bigcup_M \mathcal{Y}_M$ one can consider a *Poissonized* version of the Plancherel measure

$$\mathbb{P}_{\text{P.Pl.}}(\lambda) = \mathrm{e}^{-\theta^2} \left(\frac{\theta^{|\lambda|} F_\lambda}{|\lambda|!} \right)^2, \text{ where } |\lambda| = \text{weight}(\lambda).$$

Remark The result of B–D–J was obtained after the study of large k asymptotic behavior of the distributions $\mathbb{P}_{\text{P.Pl.}} (\lambda_1 \leq k)$, finding

$$\lim_{\theta \to \infty} \mathbb{P}_{\text{P.Pl.}} \left(\frac{\lambda_1 - 2\theta}{\theta^{1/3}} \leq s \right) = F(s)$$

and thereafter using a *de-Poissonization* procedure.

Toeplitz determinants in the story

[Gessel's formula] The distributions are given by

$$\mathbb{P}_{\text{P.Pl.}}(\lambda_1 \leq k) = \mathrm{e}^{-\theta^2} D_{k-1}(\varphi)$$

where $D_k(\varphi)$ are Toeplitz determinants associated to the symbol $\varphi = \varphi[\theta](z)$ is given by $\varphi = \mathrm{e}^{w(z)}$ for $w(z) = v(z) + v(z^{-1})$ and $v(z) = \theta z$.

In particular

$$\boxed{D_k := \det(T_k(\varphi))}$$

with $T_k(\varphi)$ being the k-th Toeplitz matrix associated to the symbol $\varphi(z)$

$$T_k(\varphi)_{i,j} := \varphi_{i-j}, \quad i,j = 0, \dots, n$$

where for every $\ell \in \mathbb{Z}$, φ_ℓ is the ℓ-th Fourier coefficient of $\varphi(z)$, namely

$$\varphi_\ell = \int_{-\pi}^{\pi} e^{-i\ell\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi}, \text{ so that } \sum_{\ell \in \mathbb{Z}} \varphi_\ell z^\ell = \varphi(z).$$

Borodin formula and its continuous limit

[Borodin, Adler - Van Moerbeke, Baik, 2000] For every $k \geq 1$ we have

$$\frac{D_k D_{k-2}}{D_{k-1}^2} = 1 - x_k^2$$

where x_k solves the so called *discrete Painlevé II* equation, which corresponds to the second order nonlinear difference equation

$$\theta(x_{k+1} + x_{k-1})(1 - x_k^2) + kx_k = 0$$

with initial conditions $x_0 = -1, x_1 = \varphi_1/\varphi_0$.

In the limit for $\theta \to \infty$ and for $k = s\theta^{1/3} + 2\theta$ (or $s = (k - 2\theta)\theta^{-1/3}$), then

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2, \qquad x_{k+1} + x_{k-1} = -\frac{kx_k}{\theta(1 - x_k^2)}$$

B–D–J $\downarrow$ $\boxed{x_k = (-1)^k \theta^{-1/3} u(s)}$ $\qquad\qquad$ $\downarrow$ $\boxed{x_k = (-1)^k \theta^{-1/3} u(s)}$

$$\partial_s^2 \log F(s) = -u^2(s), \qquad \underbrace{u''(s) = 2u^3(s) + su(s)}_{\text{Painlevé II equation}}$$

Outline

Higher order Airy kernel and free fermions

Consider a system of N free fermions in 1d with anharmonic trap ($V(x) \sim x^{2n}, n \in \mathbb{N}$).

Fact Positions/momenta in the harmonic case ($n = 1$) behave like eigenvalues of GUE.

[Le Doussal - Majumdar - Schehr, 2018] When the system is at zero temperature, they proved that

$$\lim_{N \to \infty} \mathbb{P}\left(\frac{p_{max}(N) - p_{edge}(N)}{p_N} \leq s \right) = F_n(s),$$

with $F_n(s)$ the Fredholm determinant

$$F_n(s) := \det \left(1 - \mathcal{K}^{\mathrm{Ai}_n}|_{(s,+\infty)} \right)$$

where $\mathcal{K}^{\mathrm{Ai}_n}$ is the integral operator acting through the n-th order Airy kernel

$$K_s^{\mathrm{Ai}_n}(x, y) := \int_0^{+\infty} \mathrm{Ai}_n(x + t)\mathrm{Ai}_n(y + t) dt,$$

Ai_n being the n-th Airy function

$$\mathrm{Ai}_n(x) = \frac{1}{\pi} \int_0^{+\infty} \cos\left(\frac{t^{2n+1}}{2n+1} + xt \right) dt, \;\; x \in \mathbb{R}.$$

Generalization of the Tracy-Widom formula

[Cafasso - Claeys - Girotti, 2019] The Fredholm determinant $F_n(s)$ satisfies

$$\frac{d^2}{ds^2} \ln F_n(s) = -u^2((-1)^{n+1} s)$$

where u solves the n-th member of the homogeneous Painlevé II hierarchy with boundary condition $u(s) \sim Ai_n(s)$ for $s \to +\infty$.

The Painlevé II hierarchy

The Painlevé II hierarchy is a sequence of nonlinear differential equations

$$\left(\frac{d}{ds}+2u\right)\mathcal{L}_n\left[u_s-u^2\right]=su+\alpha_n,$$

where $\mathcal{L}_n[\cdot]$ the are Lenard polynomials computed through the following recursion

$$\frac{d}{ds}\mathcal{L}_{n+1}\left[w\right]=\left(\frac{d^3}{ds^3}+4w\frac{d}{ds}+2w_s\right)\mathcal{L}_n\left[w\right],\quad n\geq 0\ \text{ with }\ \mathcal{L}_0\left[w\right]=\frac{1}{2},$$

replacing $w=u_s-u^2$.

Examples

$$\begin{aligned}
n=1: &\qquad u''-2u^3=su+\alpha_1,\\
n=2: &\qquad u''''-10u(u')^2-10u^2u''+6u^5=su+\alpha_2,\\
n=3: &\qquad u''''''-14u^2u''''-56uu'u'''-70(u')^2u''-42u(u'')^2+70u^4u''\\
&\qquad +140u^3(u')^2-20u^7=su+\alpha_3.
\end{aligned}$$

Multicritical random partitions

[Okunkov, 2001] On the set of all partitions consider the *Schur* measures

$$\mathbb{P}_{\text{Sc.}}(\lambda) = Z^{-1} s_\lambda \left[\theta_1, \dots, \theta_n\right]^2,$$

where s_λ can be computed as

$$s_\lambda \left[\theta_1, \dots, \theta_n\right] = \det_{i,j} h_{\lambda_i - i + j} \left[\theta_1, \dots, \theta_n\right],$$

with $\sum_{k \geq 0} h_k z^k = \mathrm{e}^{v(z)}$, $v(z) = \sum_{i=1}^{n} \frac{\theta_i}{i!} z^i$ and $Z = \mathrm{e}^{\sum_{i=1}^{n} \frac{\theta_i^2}{i}}$.

Remark For $n = 1$ with $\mathbb{P}_{\text{P.Pl.}}(\lambda) = \mathbb{P}_{\text{Sc.}}(\lambda)$ with $\theta_1 = \theta$.

The probability distribution of the first part of such a random partition is again given by Toeplitz determinants

$$\mathbb{P}_{\text{Sc.}}\left(\lambda_1 \leq k\right) = \mathrm{e}^{-\sum_i^n \hat{\theta}_i^2 / i} D_{k-1}(\varphi^{(n)} \left[\hat{\theta}_1, \dots, \hat{\theta}_n\right]),$$

where the symbol $\varphi^{(n)} \left[\hat{\theta}_1, \dots, \hat{\theta}_n\right](z) = \mathrm{e}^{w(z)}$, $w(z) = v(z) + v(z^{-1})$, $\theta_i \to \hat{\theta}_i = (-1)^{i+1} \theta_i$.

The result of Betea-Bouttier-Walsh

[Betea–Bouttier–Walsh , 2021] Let

$$\theta_i = (-1)^{i+1} \frac{(n-1)!(n+1)!}{(n-i)!(n+i)!} \theta = (-1)^{i+1} \hat{\theta}_i,$$

then the limiting behavior of the distribution of the first part is described by

$$\lim_{\theta \to \infty} \mathbb{P}_{\text{Sc.}} \left(\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2n+1}}} < s \right) = F_n(s),$$

where $b = \frac{n+1}{n}$, $d = \binom{2n}{n-1}^{-1}$.

The global picture

Discrete | **Continuous**

n=1

$\mathbb{P}_{\text{P.Pl.}}(\lambda_1 \leq k) = \mathrm{e}^{-\theta^2} D_{k-1}(\varphi)$ with $\varphi = \varphi^{(1)}[\theta_1 = \theta](z)$ and

$$D_{k-2}D_k / D_{k-1}^2 = 1 - x_k^2$$

with x_k solving

$$\text{dPII}: \ \theta(x_{k+1} + x_{k-1})(1 - x_k^2) + kx_k = 0.$$

$$\lim_{\theta\to\infty} \mathbb{P}_{\text{P.Pl.}}\left(\frac{\lambda_1 - 2\theta}{\theta^{\frac{1}{3}}} \leq s\right) = F(s)$$

and

$$\partial_s^2 \log F(s) = -u^2(s)$$

with u solving

$$\text{PII}: \quad u''(s) = 2u^3(s) + su(s).$$

n>1

$\mathbb{P}_{\text{Sc.}}(\lambda_1 \leq k) = \mathrm{e}^{-\sum_i^N \hat{\theta}_i^2/i} D_{k-1}(\varphi)$ with $\varphi = \varphi^{(n)}\left[\hat{\theta}_1, \dots, \hat{\theta}_n\right](z)$ and

what is the recursion relation for D_k?

$\lim_{\theta\to\infty} \mathbb{P}_{\text{Sc.}}\left(\frac{\lambda_1 - b\theta}{(d\theta)^{\frac{1}{2n+1}}} \leq s\right) = F_n(s)$ and $\partial_s^2 \log F_n(s) = -u^2((-1)^{n+1}s)$ with u solving the n-th higher order analogue of PII.

Outline

Our main result

Theorem (Chouteau - T., 2022)

For any fixed $n \geq 1$, for the Toeplitz determinants $D_k, k \geq 1$, we have

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$

where now x_k solves the $2n$ order nonlinear difference equation

$$kx_k + \left(v_k + v_k Perm_k - 2x_k \Delta^{-1}\left(x_k - (\Delta + I)x_k Perm_k\right)\right) L^n(0) = 0$$

where L is a discrete recursion operator that acts as follows

$$L(u_k) := \left(x_{k+1}\left(2\Delta^{-1} + I\right)\left((\Delta + I)\, x_k Perm_k - x_k\right) + v_{k+1}\,(\Delta + I) - x_k x_{k+1}\right) u_k,$$

and $L(0) = \theta_n x_{k+1}$. Here $v_k := 1 - x_k^2$, Δ denotes the difference operator $\Delta : u_k \to u_{k+1} - u_k$ and $Perm_k$ is the transformation

$$Perm_k : \begin{array}{ccc} \mathbb{C}\left[(x_j)_{j\in[[0,2k]]}\right] & \longrightarrow & \mathbb{C}\left[(x_j)_{j\in[[0,2k]]}\right] \\ P\left((x_{k+j})_{-k\leqslant j\leqslant k}\right) & \longmapsto & P\left((x_{k-j})_{-k\leqslant j\leqslant k}\right). \end{array}$$

The first equations of the hierarchy

$$n=1: \quad kx_k + \theta_1(x_{k+1}+x_{k-1})(1-x_k^2) = 0, \leftarrow$$ discrete Painlevé II equation

$$n=2: \quad kx_k + \theta_1(1-x_k^2)\,(x_{k+1}+x_{k-1})$$
$$+\,\theta_2(1-x_k^2)\left(x_{k+2}(1-x_{k+1}^2) + x_{k-2}(1-x_{k-1}^2) - x_k(x_{k+1}+x_{k-1})^2\right) = 0,$$

$$n=3: \quad kx_k + \theta_1(1-x_k^2)\,(x_{k+1}+x_{k-1})$$
$$+\,\theta_2(1-x_k^2)\left(x_{k+2}(1-x_{k+1}^2) + x_{k-2}(1-x_{k-1}^2) - x_k(x_{k+1}+x_{k-1})^2\right)$$
$$+\,\theta_3(1-x_k^2)\left(x_k^2(x_{k+1}+x_{k-1})^3 + x_{k+3}(1-x_{k+2}^2)(1-x_{k+1}^2) + x_{k-3}(1-x_{k-2}^2)(1-x\right.$$
$$+\,\theta_3(1-x_k^2)\left(-2x_k(x_{k+1}+x_{k-1})(x_{k+2}(1-x_{k+1}^2) + x_{k-2}(1-x_{k-1}^2))\right)$$
$$+\,\theta_3(1-x_k^2)\left(-x_{k-1}x_{k-2}^2(1-x_{k-1}^2) - x_{k+1}x_{k+2}^2(1-x_{k+1}^2)\right)$$
$$+\,\theta_3(1-x_k^2)\,(-x_{k+1}x_{k-1}(x_{k+1}+x_{k-1})) = 0.$$

Connection with orthogonal polynomials

We consider the measure for $z = \mathrm{e}^{i\alpha} \in S^1$ given by

$$\mathrm{d}\mu(\alpha) = \varphi(\mathrm{e}^{i\alpha})\frac{\mathrm{d}\alpha}{2\pi} = \mathrm{e}^{w(\mathrm{e}^{i\alpha})}\frac{\mathrm{d}\alpha}{2\pi}.$$

The family $\{p_k(z)\}_{k\in\mathbb{N}}$ of orthogonal polynomials on the unitary circle (OPUC) w.r.t. the measure is given by

$$p_k(z) = \kappa_k z^k + \dots \kappa_0, \quad \kappa_k > 0$$

such that the following relation holds for any index k, h

$$\int_{-\pi}^{\pi} \overline{p_k(e^{i\alpha})} p_h(e^{i\alpha}) \frac{\mathrm{d}\mu(\alpha)}{2\pi} = \delta_{k,h}.$$

Then for any $k \geq 1$

$$p_k(z) = \frac{1}{\sqrt{D_k D_{k-1}}} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-k+1} & \varphi_{-k} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-k+2} & \varphi_{-k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{k-1} & \varphi_{k-2} & \cdots & \varphi_0 & \varphi_{-1} \\ 1 & z & \dots & z^{k-1} & z^k \end{pmatrix}, \quad \Longrightarrow \quad \frac{D_{k-1}}{D_k} = \kappa_k^2.$$

About the proof

1. This family of orthogonal polynomials $\{p_k(z)\}$ can be characterized by a 2×2 matrix-valued Riemann–Hilbert problem (part of Baik-Deift-Johansson's results for the generalized weight).
2. From the explicit form of the solution of the Riemann–Hilbert problem, one can easly deduce the formula
$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$
where $x_k = \frac{1}{\kappa_k} p_k(0)$.
3. The solution to the Riemann–Hilbert problem allows to construct a *Lax pair* (sort of linear representation) for the discrete Painlevé II hierarchy for x_k.
4. This Lax pair is mapped into the original Lax pair obtained by Cresswell and Joshi in 1998 which first introduced the discrete Painlevé II hierarchy.

The new continuous limit

Recall the B–B–W result for $n = 2$: $\lim_{\theta\to\infty} \mathbb{P}_{\text{Sc.}}\left(\frac{\lambda_1 - \frac{3}{2}\theta}{(4^{-1}\theta)^{1/5}} \leq s\right) = F_2(s)$.

In the limit for $\theta \to \infty$, taking $k = s\left(\frac{\theta}{4}\right)^{1/5} + \frac{3}{2}\theta$ (or $s = \left(k - \frac{3}{2}\theta\right)\theta^{-\frac{1}{5}}4^{\frac{1}{5}}$)

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2,$$

$$\text{B–B–W}\downarrow \quad \boxed{x_k = (-1)^k \left(\frac{\theta}{4}\right)^{-1/5} u(s)}$$

$$\partial_s^2 \log F_2(s) = -u^2(s),$$

$$k x_k + \theta_1 v_k \left(x_{k+1} + x_{k-1}\right) + \theta_2 v_k \left(x_{k+2} v_{k+1} + x_{k-2} v_{k-1} - x_k (x_{k+1} + x_{k-1})^2\right) = 0$$

$$\downarrow \quad \boxed{x_k = (-1)^k \left(\frac{\theta}{4}\right)^{-1/5} u(s) \;\; \theta_1 = \theta,\;\; \theta_2 = \frac{\theta}{4}}$$

$$\underbrace{u'''' - 10u(u')^2 - 10u^2 u'' + 6u^5 = -su}_{\text{2nd eq. of the Painlevé II hierarchy}}$$

which recovers the generalized Tracy-Widom formula for the higher order Airy kernels [Cafasso–Claeys–Girotti, 2019] for $n = 2$. And so on ...

Remark This gives an alternative continuous limit w.r.t. the classical one (proposed by Cresswell – Joshi, 1998).

Thank you!