Integrability of deformed sine kernel determinants

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Outline

Background on sine vs Airy kernels

The sine and Airy kernels are functions of two variables $(x, y) \in \mathbb{R}^2$ defined respectively as

$$K^{\sin}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}, \quad \text{and} \quad K^{\mathrm{Ai}}(x,y) = \int_0^{+\infty} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\mathrm{d}u.$$

The corresponding integral operators are denoted by $\mathcal{K}^{\sin}$ and $\mathcal{K}^{\mathrm{Ai}}$.

[Soshnikov, 2000] Hermitian locally trace class operator $\mathcal{K}$ on $L^2(\mathbb{R})$ with kernel $K(\cdot,\cdot)$ defines a determinantal point process on $\mathbb{R}$ with

$$\rho_k(\xi_1, \ldots, \xi_k) = \det_{i,j=1}^{k} \left(K(\xi_i, \xi_j)\right)$$

if and only if $0 \le \mathcal{K} \le 1$. If the corresponding point process exists it is unique.

$$\downarrow$$

$\mathcal{K}^{\sin}$ and $\mathcal{K}^{\mathrm{Ai}}$ define the sine and Airy DPPs on $\mathbb{R}$.

GUE: bulk vs edge behavior

The Gaussian Unitary Ensemble is built up by taking the set of Hermitian $N \times N$ matrices, together with

$$\mathbb{P}(M)dM = \frac{1}{Z_N} e^{-\operatorname{tr} M^2/2} dM,$$

$$\downarrow$$

The eigenvalues of GUE matrices are described by a determinantal point process where the correlation kernel K_N is written in terms of Hermite polynomials.

In the large N limit, the behavior of the eigenvalues' correlation kernels is

Bulk

$$(Nd(x_0))^{-1} K_N \left(x_0 + \frac{x}{Nd(x_0)}, x_0 + \frac{y}{Nd(x_0)} \right)$$

$$\downarrow \; N \to \infty$$

$$K^{\sin}(x, y)$$

Edge

$$N^{-2/3} K_N \left(2 + \frac{x}{N^{2/3}}, 2 + \frac{y}{N^{2/3}} \right)$$

$$\downarrow \; N \to \infty$$

$$K^{\mathrm{Ai}}(x, y)$$

Their common structures

- They are both of integrable IIKS type

$$K(x,y)=\frac{\vec{f}^{\top}(x)\vec{g}(y)}{x-y},\quad \text{with}\quad \vec{f}^{\top}(x)\vec{g}(x)=0.$$

In particular

$$K^{\sin}(x,y)=\frac{\mathrm{e}^{i\pi x}\mathrm{e}^{-i\pi y}-\mathrm{e}^{-i\pi x}\mathrm{e}^{i\pi y}}{2\pi i(x-y)},\qquad K^{\mathrm{Ai}}(x,y)=\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

- They both have integral representations

$$K^{\sin}(x,y)=\int_{-1/2}^{1/2}\mathrm{e}^{2\pi ixu}\mathrm{e}^{-2\pi iyu}du\qquad K^{\mathrm{Ai}}(x,y)=\int_0^{+\infty}\mathrm{Ai}(x+u)\mathrm{Ai}(y+u)du.$$

- Thus their integral operators can both be decomposed as

$$\mathcal{K}^{\sin}=\mathcal{F}^*\chi_{(-1/2,1/2)}\mathcal{F}$$

for $\mathcal{F}$ being the Fourier transform and $\chi_{(-1/2,1/2)}$ the projection on the interval $(-1/2,1/2)$.

$$\mathcal{K}^{\mathrm{Ai}}=\mathcal{A}\chi_{(0,\infty)}\mathcal{A}^*$$

for $\mathcal{A}$ being the Airy transform and $\chi_{(0,\infty)}$ the projection on the interval $(0,\infty)$.

The JMMS and TW formulas

Of particular interest for the sine and Airy DPPs are the following gap probabilities expressed in terms of Fredholm determinants

$$F(s) = \det\left(1 - \mathcal{K}^{\sin}|_{(-s,s)}\right) \quad \text{and} \quad G(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}|_{(s,\infty)}).$$

[Jimbo-Miwa-Mori-Sato, 1980]

The Fredholm determinant $F(s)$ satisfies

$$F(s) = \exp\left(\int_0^{\pi s} \frac{\nu(x)}{x} \mathrm{d}x\right),$$

where ν is a solution of the Painlevé V σ-form equation, i.e. it solves

$$(x\nu'')^2 + 4(x\nu' - \nu)(x\nu' - \nu + (\nu')^2) = 0,$$

together with boundary condition

$$\nu(x) = -\frac{1}{\pi}x + O(x^2), \qquad x \to 0.$$

[Tracy-Widom, 1994]

The Fredholm determinant $G(s)$ satisfies

$$G(s) = \exp\left(-\int_s^{+\infty} (r-s)u^2(r)dr\right)$$

where u is the Hastings-McLeod solution of the Painlevé II equation, i.e. it solves

$$u''(s) = su(s) + 2u^3(s)$$

together with the boundary condition

$$u(s) \sim \mathrm{Ai}(s) \text{ for } s \to +\infty.$$

Deformation of the Airy kernel

For a weight function $w : \mathbb{R} \to [0, 1]$ in a certain class, $s \in \mathbb{R}$, a deformation of the Airy kernel is built as

$$K^{\mathrm{Ai}}_{w,s}(x, y) = \int_{-\infty}^{+\infty} w(z)\mathrm{Ai}(x + s + z)\mathrm{Ai}(y + z + s)dz.$$

For some specific choice of w this is often called the *finite temperature* Airy kernel.

The associated integral operator $\mathcal{K}^{\mathrm{Ai}}_{w,s}$ is s.t. $G_w(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{w,s}|_{(0,+\infty)})$ is well defined.

Remark [Amir - Corwin - Quastel, 2011] The probability distribution function of the Hopf-Cole solution $h = h(X, T) = -\log Z(X, T)$ of the KPZ equation

$$\partial_T h = \frac{1}{2}\partial_X^2 h - \frac{1}{2}(\partial_X h)^2 + \xi, \quad \xi = \xi(X, T) \text{ is a Gaussian space-time white noise}$$

with narrow wedge initial condition $Z(X, 0) = \delta_0(X)$ is written in terms of $G_w(s)$, for $w(z) = w_{KPZ}(z) = \frac{y}{y - e^{-(T/2)^{1/3} z}}$.

Integrability results

1. [Amir - Corwin - Quastel, 2011] A generalization of the Tracy–Widom formula for $G_w(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{w,s}|_{(0,+\infty)})$ (with $w = w_{KPZ}$), which reads as

$$\frac{d^2}{ds^2} \ln G_w(s) = -\int_{\mathbb{R}} \varphi^2(r;s) w'(r) dr$$

where φ solves the integro-differential Painlevé II equation

$$\frac{\partial^2}{\partial s^2} \varphi(z;s) = \left(z + s + 2 \int_{\mathbb{R}} \varphi^2(r;s) w'(r) dr \right) \varphi(z;s).$$

with $\varphi(z;s) \sim \mathrm{Ai}(z+s)$ for $s \to +\infty$ pointwise in z.

Remark For $w = \chi_{(0,+\infty)}$ one gets back $\varphi(0;s) = u(s)$ and the TW formula.

2. [Cafasso - Claeys - Ruzza, 2021] The function $u_w(x,t) = \partial_x^2 \log G_w(x,t) + \frac{x}{2t}$ for $G_w(x,t)$ the deformed Fredholm determinant associated to $t^{2/3} K^{\mathrm{Ai}}_{w_t, xt^{-1}}(t^{2/3}\cdot, t^{2/3}\cdot)$, with $w_t(\lambda) = w(t^{2/3}\lambda)$, solves the Korteweg-de Vries equation

$$\partial_t u_w + 2 u_w \partial_x u_w + \frac{1}{6} \partial_x^3 u_w = 0.$$

Other appearences

The deformed Airy kernel together with the analogue deformed sine kernel have been found in multiple models for specific choices of the weight function w.

- [Johansson, 2008, Lietchy - Wang, 2018] Here $w(r) = (1 + e^{-\alpha r})^{-1}$ $w(r) = (1 + e^{-\lambda(4r^2-1)})^{-1}$. The limiting behavior of the eigenvalues in the Moshe-Neurberg-Shapiro model in the bulk / edge.
- [Dean - Ledoussal - Majumdar - Schehr, 2018] Here w is essentially the same as above (but α, λ are proportional to the inverse temperature). The limiting behavior of the positions of a system of free fermions at finite temperature trapped with certain class of potentials ($V(x) \sim x^{2n}$) in the bulk / edge.
- [Bothner - Little, 2022] Here $w(r) = \Phi(s\sigma^{-1}(r+1)) - \Phi(s\sigma^{-1}(r-1))$ with $\Phi(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} \mathrm{e}^{-y^2} dy$. The limiting (weak non-hermiticity limit) behavior of the real parts of eigenvalues of the Complex Elliptic Ginibre ensemble in the bulk / edge.

Deformations of the sine kernel

For any integrable function $w : \mathbb{R} \to [0,1]$, we consider the deformed sine kernel

$$K_w^{\sin}(x,y) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i(x-y)u} w(u)\mathrm{d}u.$$

The Fredholm determinant $F_w(s) = \det\left(1 - \mathcal{K}_w^{\sin}|_{(-s,s)}\right)$ associated to the integral operator $\mathcal{K}_w^{\sin}$ is well-defined, and we have

$$F_w(s) = \det\left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right),$$

where $\sqrt{w_s}$ denotes the multiplication operator with a square root of the function $w(\frac{\cdot}{2s})$.

Remark Writing $F_w(s)$ in this form we can use the Riemann–Hilbert problem associated to the integrable IIKS structure

$$\vec{f}(x) = \frac{\sqrt{w_s(x)}}{2\pi i}\begin{pmatrix} \mathrm{e}^{i\pi x} \\ \mathrm{e}^{-i\pi x}\end{pmatrix}, \quad \vec{g}(y) = \sqrt{w_s(y)}\begin{pmatrix} \mathrm{e}^{-i\pi y} \\ -\mathrm{e}^{i\pi y}\end{pmatrix}.$$

Aim

- Generalization of JMMS formula for $F_w(s)$;
- Suitable deformation which relates it to an integrable PDE.

Outline

Main result

Theorem (Claeys - T.)

For every $s > 0$, we have the identity

$$\partial_s s \partial_s \log F_w(s) = 2 \int_{\mathbb{R}} \lambda w'(\lambda) \phi(\lambda, s) \psi(\lambda, s) \mathrm{d}\lambda,$$

where ϕ, ψ solve the (Zakharov-Shabat) system of equations

$$\partial_s \phi(\lambda, s) = 2i\pi \left(\lambda \phi(\lambda, s) + \frac{1}{4\pi^2 s} \int_{\mathbb{R}} \phi^2(\mu, s) w'(\mu) \mathrm{d}\mu \, \psi(\lambda, s) \right),$$

$$\partial_s \psi(\lambda, s) = -2i\pi \left(\frac{1}{4\pi^2 s} \int_{\mathbb{R}} \psi^2(\mu, s) w'(\mu) \mathrm{d}\mu \, \phi(\lambda, s) + \lambda \psi(\lambda, s) \right).$$

with $\lambda \to \infty$ asymptotics $\phi(\lambda, s) \sim \mathrm{e}^{2\pi i s \lambda}$, $\psi(\lambda, s) \sim \mathrm{e}^{-2\pi i s \lambda}$.

Remark If w is even, then

$$\partial_s s \partial_s \log F_w(s) = 2 \int_{\mathbb{R}} \lambda w'(\lambda) \phi(\lambda, s) \phi(-\lambda, s) \mathrm{d}\lambda,$$

where ϕ solves the integro-differential equation

$$\partial_s \phi(\lambda, s) = 2\mathrm{i}\pi \left(\lambda \phi(\lambda, s) + \frac{1}{4\pi^2 s} \int_{\mathbb{R}} \phi^2(\mu, s) w'(\mu) \mathrm{d}\mu \, \phi(-\lambda, s) \right), \; \phi(\lambda, s) \sim_{\lambda \to \infty} \mathrm{e}^{2\pi \mathrm{i} s \lambda}.$$

Reduction to Painlevé V

Back to the *zero temperature case* for $w(r) = \chi_{(-\frac{1}{2},\frac{1}{2})}$ then $w'(r) = \delta_{-\frac{1}{2}}(r) - \delta_{\frac{1}{2}}(r)$.

Then our integro-differential equations reduce to

$$\partial_s \phi\left(\pm\frac{1}{2}, s\right) = 2i\pi\left(\pm\frac{1}{2}\phi\left(\pm\frac{1}{2}, s\right) + \frac{1}{4\pi^2 s}\left(\phi^2\left(-\frac{1}{2}, s\right) - \phi^2\left(\frac{1}{2}, s\right)\right)\phi\left(\mp\frac{1}{2}, s\right)\right)$$

and by defining

$$v(x) = \frac{1}{\pi \mathrm{i}}\phi\left(\frac{1}{2}, \frac{x}{2\pi \mathrm{i}}\right)\phi\left(-\frac{1}{2}, \frac{x}{2\pi \mathrm{i}}\right), \qquad u(x) = \frac{\phi^2\left(\frac{1}{2}, \frac{x}{2\pi \mathrm{i}}\right)}{\phi^2\left(-\frac{1}{2}, \frac{x}{2\pi \mathrm{i}}\right)},$$

we recover the system

$$xv' = v^2(u - \frac{1}{u}), \quad xu' = xu - 2v(u-1)^2$$

implying that u solves the Painlevé V equation

$$u'' = \frac{u}{x} - \frac{u'}{x} - \frac{u(u+1)}{2(u-1)} + (u')^2\frac{3u-1}{2u(u-1)}.$$

Moreover, the JMMS formula is recovered by $\nu'(s) = v(2\mathrm{i}s) = -\frac{2}{\pi}\phi\left(\frac{1}{2}, \frac{s}{\pi}\right)\phi\left(-\frac{1}{2}, \frac{s}{\pi}\right)$.

Alternative integro-differential system

We can also obtain an analogue integro-differential system for

$$v(\lambda, s) = \frac{1}{2\pi\mathrm{i}}\phi\left(\lambda, \frac{s}{2\pi\mathrm{i}}\right)\phi\left(-\lambda, \frac{s}{2\pi\mathrm{i}}\right), \quad u(\lambda, s) = \frac{\phi^2\left(\lambda, \frac{s}{2\pi\mathrm{i}}\right)}{\phi^2\left(-\lambda, \frac{s}{2\pi\mathrm{i}}\right)}.$$

In particular

$$\partial_s v(\lambda, s) = -\frac{v(\lambda, s)}{s}\left(u^{\frac{1}{2}}(\lambda, s) + u^{-\frac{1}{2}}(\lambda, s)\right)\int_{\mathbb{R}} v(\lambda, s)u^{\frac{1}{2}}(\lambda, s)w'(\lambda)\mathrm{d}\lambda,$$
$$\partial_s u(\lambda, s) = 2\lambda u(\lambda, s) - \frac{2}{s}\left(u^{\frac{1}{2}}(\lambda, s) - u^{\frac{3}{2}}(\lambda, s)\right)\int_{\mathbb{R}} v(\lambda, s)u^{\frac{1}{2}}(\lambda, s)w'(\lambda)\mathrm{d}\lambda.$$

Remark From this system we can derive an integro-differential second order (in s) equation for $u(\lambda, s)$, which degenerates to the Painlevé V equation when w is the characteristic function, but where some parts still depend on $v(\lambda, s)$.

Relation with other integro-differential equation

[Bothner - Little, 2022] For the same Fredholm determinant $F_w(s)$ another characterization is given

$$\frac{d^2}{ds^2}\log F_w(s) = \left(\int_0^\infty r(s,\lambda)w'(\lambda)d\lambda\right)^2$$

for $r(s,\lambda)$ a solution of another (new) integro-differential equation.

This equation reduces, in the case of $w = \chi_{(-1/2,1/2)}$, to a differential equation for $r(s, 1/2)$ which is known to be related to the Painlevé V σ-form by the identity

$$\frac{d}{ds}\left(\frac{\nu(s)}{s}\right) = -r^2(s,1/2).$$

Remark How the equation for $r(s,\lambda)$ is related to the system of equations for $u(\lambda, s)$, $v(\lambda, s)$ is to be understood.

RH problem

$U(\lambda; s) = U^{(w)}(\lambda; s) : \mathbb{C} \to GL(2, \mathbb{C})$ is such that

(1) $U(\lambda; s)$ is analytic for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(2) $U(\lambda; s)$ has continuous boundary values $U_\pm(\lambda; s)$ when λ approaches $\mathbb{R}$ from either above $(+)$ or below $(-)$ and they satisfy the jump condition

$$U_+(\lambda; s) = U_-(\lambda; s) \underbrace{\begin{pmatrix} 1 & 1 - w(\lambda) \\ 0 & 1 \end{pmatrix}}_{= J_U(\lambda)}, \quad \lambda \in \mathbb{R}.$$

(3) There exists a matrix $U_1 = U_1(s)$ such that we have as $\lambda \to \infty$,

$$U(\lambda; s) = \left(I_2 + \frac{U_1}{\lambda} + O(\lambda^{-2}) \right) e^{2\pi i s \lambda \sigma_3} \begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & \operatorname{Im} \lambda > 0, \\ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} & \operatorname{Im} \lambda < 0, \end{cases}$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The road to the coupled system

Proposition

The unique solution $U(\lambda; s)$ to the RH problem for U solves the following linear differential equation,

$$\partial_s U(\lambda; s) = M(\lambda, s) U(\lambda; s), \quad M(\lambda; s) = 2\mathrm{i}\pi \begin{pmatrix} \lambda & -2\beta(s) \\ 2\gamma(s) & -\lambda \end{pmatrix},$$

where

$$\beta(s) := [U_1(s)]_{1,2}, \quad \gamma(s) := [U_1(s)]_{2,1},$$

Moreover, if w is even,

$$\beta(s) = -\gamma(s), \qquad \partial_s \alpha = 4\pi\mathrm{i}\gamma^2, \quad \text{for } \alpha(s) = [U_1(s)]_{1,1}.$$

In addition, for $\phi(\lambda; s) := U_{1,1}(\lambda; s)$ and $\psi(\lambda; s) := U_{2,1}(\lambda; s)$, we have the following *trace formulas*

$$\beta(s) = -\frac{1}{8\pi^2 s} \int_{\mathbb{R}} \phi^2(\lambda; s) w'(\lambda) \mathrm{d}\lambda, \quad \gamma(s) = -\frac{1}{8\pi^2 s} \int_{\mathbb{R}} \psi^2(\lambda; s) w'(\lambda) \mathrm{d}\lambda.$$

together with the orthogonality condition

$$\int_{\mathbb{R}} \phi(\lambda; s) \psi(\lambda; s) w'(\lambda) \mathrm{d}\lambda = 0.$$

And the road to the Fredholm determinant formula

Proposition

Let $w : \mathbb{R} \to [0,1]$ *be integrable and* C^1. *Then the unique solution of the RH problem* $U^{(w)}$ *is such that*

$$\partial_s \log F_w(s) = \frac{1}{2\pi i s} \int_{\mathbb{R}} \left[U_+^{(w)}(r;s)^{-1} \frac{\mathrm{d}}{\mathrm{d}r} U_+^{(w)}(r;s) \right]_{2,1} w'(r) r \mathrm{d}r.$$

This result is rewritten as

$$\partial_s \log F_w(s) = \frac{1}{2\pi i s} \int_{\mathbb{R}} r w'(r) (\psi(r;s) \partial_r \phi(r;s) - \phi(r,s) \partial_r \psi(r;s)) \mathrm{d}r.$$

Together with the simplification

$$\partial_s (\psi(r,s) \partial_r \phi(r,s) - \phi(r,s) \partial_r \psi(r,s)) = 4\pi i \phi(r,s) \psi(r,s),$$

the final result follows

$$\partial_s s \partial_s \log F_w(s) = 2 \int_{\mathbb{R}} r w'(r) \phi(r,s) \psi(r,s) \mathrm{d}r.$$

Outline

The new t parameter

Let $W : \mathbb{R} \to [0,1]$ be C^1 and such that $W(.^2 - t)$ is integrable on $[0, +\infty)$ for any $t \in \mathbb{R}$. Then, we define a family of even functions w_t, as

$$w_t(u) = W(u^2 - t), \qquad t \in \mathbb{R},$$

and consider the Fredholm determinant

$$Q_W(s,t) := \det\left(1 - \mathcal{K}^{\sin}_{w_t}|_{(-s,s)}\right) = F_{w_t}(s).$$

Remark An alternative expression for Q_W is

$$Q_W(s,t) = \det\left(1 - \mathcal{K}^{\sin}_{w_{s,t}}|_{(-1/2,1/2)}\right),$$

where

$$w_{s,t}(r) = w_t\left(\frac{r}{2s}\right) = W\left(\frac{r^2}{4s^2} - t\right).$$

Moreover, as before, we also have the fundamental identity

$$Q_W(s,t) := \det\left(1 - \sqrt{w_{s,t}}\mathcal{K}^{\sin}\sqrt{w_{s,t}}\right),$$

and thus the results on the s-dependence of $Q_W(s,t)$ hold identically as before for $F_{w_{s,t}}$.

The main result

Define $\sigma_W(x,t)$ and $q_W(x,t)$ by

$$\sigma_W(x,t) := \log Q_W(s = \frac{x}{\pi}, t), \qquad q_W(x,t)^2 = -\partial_x^2 \sigma(x,t).$$

Theorem (Claeys, T.)

For any $x > 0$, $t \in \mathbb{R}$, $q = q_W$ solves the PDE

$$\partial_x \left(\frac{\partial_x \partial_t q}{2q} \right) = \partial_t (q^2) - 1,$$

and $\sigma = \sigma_W$ solves the PDE

$$(\partial_x^2 \partial_t \sigma)^2 = 4\partial_x^2 \sigma \left(-2x \partial_x \partial_t \sigma + 2\partial_t \sigma - (\partial_x \partial_t \sigma)^2 \right).$$

Remark For $w_t(u) = W(u^2 - t)$ and $W(y) = \frac{1}{e^{4y}+1}$, the connection between $Q_W(s,t) = \det\left(1 - \sqrt{w_{s,t}} \mathcal{K}^{\sin} \sqrt{w_{s,t}}\right)$ and these PDEs was already found in [Its - Izergin - Korepin - Slavnov, 1990] in the study of the impenetrable one dimensional Bose gas.

The second Lax equation

We consider the RH problem for U corresponding to $w = w_t$ so that

$$U(\lambda; s, t) := U^{(w_t)}(\lambda; s)$$

and we introduce the differential operator

$$D_{\lambda,t} = \partial_\lambda + 2\lambda\partial_t,$$

that combines the dependence on the spectral parameter λ and the deformation parameter t.

Remark We recall that the jump matrix of $U^{(w_t)}$ is given by

$$\begin{pmatrix} 1 & 1 - W(\lambda^2 - t) \\ 0 & 1 \end{pmatrix},$$

and $D_{\lambda,t} W(\lambda^2 - t) = 0$.

Proposition

We have $D_{\lambda,t} U(\lambda, s, t) = L(\lambda, s, t) U(\lambda, s, t)$, *with*

$$L(\lambda, s, t) = \begin{pmatrix} 2\pi \mathrm{i} s + 2\partial_t \alpha(s,t) & -2\partial_t \gamma(s,t) \\ 2\partial_t \gamma(s,t) & -2\pi \mathrm{i} s - 2\partial_t \alpha(s,t) \end{pmatrix}.$$

The compatibility condition

We end up with the system for U given by

$$\partial_s U(\lambda,s,t) = M(\lambda,s,t)U(\lambda,s,t) \quad D_{\lambda,t}U(\lambda,s,t) = L(\lambda,s,t)U(\lambda,s,t)$$

for which the compatibility condition reads as

$$\partial_s L - D_{\lambda,t}M + [L,M] = 0,$$

and is equivalent to a coupled system of PDEs for $\alpha(s,t)$ and $\gamma(s,t)$, namely

$$\begin{aligned} -8\mathrm{i}\pi\gamma\partial_t\gamma + \partial_t\partial_s\alpha &= 0,\\ 8\pi\gamma(\pi s - \mathrm{i}\partial_t\alpha) + \partial_t\partial_s\gamma &= 0. \end{aligned}$$

Remark

- The first equation is the t-derivative of $\partial_s\alpha = 4\pi\mathrm{i}\gamma^2$, already known.
- The second one, after taking another s-derivative and doing some manipulations, changing variable $x = 2\pi s$ and setting

  $$q(x,t) = 2\mathrm{i}\gamma\left(\frac{x}{2\pi}, t\right),$$

  gives exactly the q-PDE.

The σ-PDE

Define $p(x,t) = -\partial_x \sigma(x,t)$, so that $q^2(x,t) = -\partial_x^2 \sigma(x,t) = \partial_x p(x,t)$. We have

$$q^2(x,t) = -4\gamma^2\left(\frac{x}{2\pi}, t\right), \qquad p(x,t) = 2\mathrm{i}\alpha\left(\frac{x}{2\pi}, t\right).$$

We can thus re-write the PDE as

$$\partial_x \partial_t q = -2xq + 2q\partial_t p.$$

Multiplying with $\partial_t q$, we find

$$\frac{1}{2}\partial_x \left((\partial_t q)^2 \right) = x\partial_x^2 \partial_t \sigma + \partial_x \partial_t \sigma \; \partial_x^2 \partial_t \sigma.$$

Integrating in x, we get

$$\frac{1}{2}(\partial_t q)^2 = x\partial_x \partial_t \sigma - \partial_t \sigma + \frac{1}{2}\left(\partial_x \partial_t \sigma\right)^2.$$

Noticing that $\frac{(\partial_t q)^2}{2} = \frac{\left(\partial_t (q^2)\right)^2}{8q^2} = -\frac{\left(\partial_x^2 \partial_t \sigma\right)^2}{8\partial_x^2 \sigma}$, one can finally obtain the PDE for σ.

Remark The analogue result in the Airy case involves the KdV bilinear equation.

Formula for the Fredholm determinant

The relation between $q(x,t)$ and the Fredholm determinant, given by

$$\partial_x^2 \log Q_W(\frac{x}{2\pi}, t) = -q(x,t)^2, \quad x > 0,$$

can be deduced from the previous result

$$\partial_s s \partial_s \log Q_W(s,t) = 2\int_{\mathbb{R}} \lambda w_t'(\lambda)\phi(\lambda,s,t)\phi(-\lambda,s,t)\mathrm{d}\lambda.$$

Indeed, by residue computation we obtain

$$-\int_{\mathbb{R}} \lambda\phi(\lambda)\phi(-\lambda)w_t'(\lambda)\mathrm{d}\lambda = 2\pi\mathrm{i}\left(\alpha + 4\pi\mathrm{i}s\gamma^2\right) = 2\pi\mathrm{i}\left(\alpha + s\partial_s\alpha\right) = 2\pi\mathrm{i}\partial_s(s\alpha),$$

which after integration and derivation gives

$$\partial_s^2 \log Q_W(s,t) = 16\pi^2\gamma^2(s,t).$$

and after the change of variables this corresponds to the result for q.

Solution of the σ-PDE initial boundary value problem

The function W can be seen as scattering data for the solution $\sigma = \sigma_W(x,t)$ to the σ-PDE, in the following sense.

The solution of the σ-PDE with initial data

$$\lim_{x\to 0} \frac{1}{x}\sigma(x,t) =: F(t), \qquad t > -T,$$

where $F \in C^1(-T,+\infty)$, $F'(t)/t \in L^1(-T,+\infty)$, is
$\sigma(x,t) = \log\det\left(1 - \sqrt{w_{s,t}}\mathcal{K}^{\sin}\sqrt{w_{s,t}}\right)$ for $x = 2\pi s > 0$ and $t < T$, with

$$w_{s,t}(u) = -\frac{\pi}{2}(\mathcal{A}^{-1}F_T)\left(\sqrt{\frac{u^2}{4s^2} - t + T}\right), \quad \text{where}$$

- $F_T(s) = F(T - s^2)$;
- $\mathcal{A}$ and $\mathcal{A}^{-1}$ denote the Abel transform and its inverse, respectively

$$\mathcal{A}f(y) = 2\int_y^{+\infty} \frac{f(r)r}{\sqrt{r^2 - y^2}}\mathrm{d}r, \qquad y \in \mathbb{R},$$

  for C^1-functions $f : \mathbb{R} \to C$ with $\lim_{r\to+\infty} rf(r) = 0$ and f is integrable and

$$\mathcal{A}^{-1}F(r) = -\frac{1}{\pi}\int_r^{+\infty} \frac{F'(y)}{\sqrt{y^2 - r^2}}\mathrm{d}y, \qquad r \in \mathbb{R}.$$

Final remarks

- The final result comes from the fact that for any $T > t$ we have

$$\lim_{x\to 0}\frac{1}{x}\sigma_W(x,t) = -\frac{4}{\pi}\int_{\sqrt{T-t}}^{\infty}\frac{w_T(r)r}{\sqrt{r^2-y^2}}\mathrm{d}r = -\frac{2}{\pi}(\mathcal{A}w_T)(\sqrt{T-t}),$$

  a *simple* transformation between the initial data and the scattering data (coming from the *simple* asymptotics of the Fredholm determinant itself).

- This could in effect suggests that perhaps we should look for a PDE for $\frac{1}{x}\sigma_W(x,t)$ directly... to check!

Thank you!