Finite temperature higher order Airy kernels and an integro-differential Painlevé II hierarchy

Sofia Tarricone

IRMP, Université Catholique de Louvain-la-Neuve

Integrable Systems and Random Matrix Theory Seminar

University of Michigan, 21 March 2022

Based on the work: AIHP2101-015R1A0 with T. Bothner and M. Cafasso.

Plan

1. Introduction
   - The Painlevé II hierarchy
   - Higher order Airy kernels

2. Our object of study
   - Their finite temperature version
   - The analogue Tracy-Widom formula

3. Main ideas for the proof
   - Manipulating the kernels
   - Construction of operator-valued RH problem
   - The derivation of the operator-valued Lax pair
   - Conclusion

Outline

1. Introduction
   - The Painlevé II hierarchy
   - Higher order Airy kernels
2. Our object of study
   - Their finite temperature version
   - The analogue Tracy-Widom formula
3. Main ideas for the proof
   - Manipulating the kernels
   - Construction of operator-valued RH problem
   - The derivation of the operator-valued Lax pair
   - Conclusion

The Painlevé II equation

The *Painlevé II equation* is the nonlinear differential equation

$$\mathrm{PII}[\alpha]: \quad u_{tt} = 2u^3 + ut + \alpha.$$

A self-similarity reduction of mKdV

It can be found from the modified KdV equation

$$w_s + w_{xxx} - 6w^2 w_x = 0$$

while looking for self-similar solutions, i.e.

$$w(s,x) = \frac{u(t)}{(3s)^{\frac{1}{3}}}, \quad \text{with} \quad t = \frac{x}{(3s)^{\frac{1}{3}}}.$$

The modified KdV hierarchy and its isospectral Lax pair	$\longrightarrow$ self-similarity reduction $\longrightarrow$	The PII hierarchy and its isomonodromic Lax pair

The Painlevé II hierarchy

The n-th member of the *Painlevé II hierarchy* can be written as

$$\mathrm{PII}^{(n)}[\alpha_n]: \quad \left(\frac{d}{dt} + 2u\right)\mathcal{L}_n\left[u_t - u^2\right] = tu + \alpha_n,$$

where the Lenard polynomials $\mathcal{L}_n[w]$ are computed through the following recursion

$$\frac{d}{dt}\mathcal{L}_{n+1}[w] = \left(\frac{d^3}{dt^3} + 4w\frac{d}{dt} + 2w_t\right)\mathcal{L}_n[w], \quad n \geq 0 \text{ with } \mathcal{L}_0[w] = \frac{1}{2},$$

replacing $w = u_t - u^2$.

Examples

$$n = 1: \qquad u'' - 2u^3 = tu + \alpha_1,$$

$$n = 2: \qquad u'''' - 10u(u')^2 - 10u^2u'' + 6u^5 = tu + \alpha_2,$$

$$n = 3: \qquad u'''''' - 14u^2u'''' - 56uu'u''' - 70(u')^2u'' - 42u(u'')^2 + 70u^4u'' + 140u^3(u')^2 - 20u^7 = tu + \alpha_3.$$

Here we abbreviate with $'$ the derivation w.r.t. t.

Isomonodromic representation of the Painlevé II hierarchy

[Flaschka - Newell, 1980] The Painlevé II equation admits the *isomonodromic Lax pair* given by

$$\frac{\partial \Psi}{\partial \lambda} = M\Psi, \quad \text{with} \quad M = -i(4\lambda^2 + t + 2u^2)\sigma_3 + \left(4\lambda u + \frac{\alpha}{\lambda}\right)\sigma_1 - 2u_t\sigma_2,$$
$$\frac{\partial \Psi}{\partial t} = L\Psi, \quad \text{with} \quad L = -i\lambda\sigma_3 + u\sigma_1.$$

[Clarkson - Joshi - Mazzocco, 2006] Each n-th member of the Painlevé II hierarchy admits the *isomonodromic Lax pair* given by the same system above but with

$$M \hookrightarrow M^{(n)} = \left(\sum_{j=0}^{2n} A_j (i\lambda)^j - it\right)\sigma_3 + \sum_{j=0}^{2n-1} \left(B_j\sigma_+ + C_j\sigma_-\right)(i\lambda)^j + \frac{\alpha_n}{\lambda}\sigma_1,$$

with A_j, B_j, C_j that are differential polynomials in u defined through closed formulae involving the Lenard polynomials.

This means that, for every n, the compatibility condition

$$\boxed{\frac{\partial M^{(n)}}{\partial t} - \frac{\partial L}{\partial \lambda} + M^{(n)}L - LM^{(n)} = 0 \text{ is equivalent to } \mathrm{PII}^{(n)}[\alpha_n].}$$

Outline

1. Introduction
   - The Painlevé II hierarchy
   - Higher order Airy kernels
2. Our object of study
   - Their finite temperature version
   - The analogue Tracy-Widom formula
3. Main ideas for the proof
   - Manipulating the kernels
   - Construction of operator-valued RH problem
   - The derivation of the operator-valued Lax pair
   - Conclusion

Higher order Airy kernels

Definition

Consider the integral operators $\mathcal{K}_{t,n}$ acting on $L^2(\mathbb{R}_+)$ through the higher order Airy kernels

$$K_{t,n}(x,y) := \int_{\mathbb{R}_+} Ai_n(x+z+t)Ai_n(y+z+t)dz, \quad t \in \mathbb{R}$$

where $Ai_n(t)$ is a particular solution of $\frac{d^{2n}\phi}{dt^{2n}}(t) = (-1)^{n+1}t\phi(t)$.

The Fredholm determinant $F_n(t) := \det(1 - \mathcal{K}_{t,n})$ is in relation with a class of higher order PII transcendents.

Occurences of $F_n(t)$

* **Determinantal point processes.**
  [Soshnikov, 2000] F_n is the probability distribution of the largest particle of the determinantal point process defined through n-th order Airy kernel.

* **Random matrices.**
  [Tracy - Widom, 1990] $F_1 = F_{GUE}$, is the edge scaling limit of the probability distribution of the largest eigenvalue in the Gaussian Unitary Ensemble.

* **Fermionic systems at zero temperature.**
  [LeDoussal - Majumdar - Schehr, 2018] F_n is the probability distribution in the limit near the edge of the rescaled largest momentum of a system of free fermions in anharmonic potential at zero temperature.

* **Random partitions.**
  [Betea - Bouttier - Walsh, 2021] F_n describes certain limiting behavior at the edge of multricritial Schur measures.

Their relation with a class of Painlevé II transcendents

[Ledoussal - Majumdar - Schehr, 2018] ; [Cafasso - Claeys - Girotti, 2019]
For every $(t, n) \in \mathbb{R} \times \mathbb{N}$, the Fredholm determinant $F_n(t)$ satisfies

$$\frac{d^2}{dt^2} \ln F_n(t) = -u^2((-1)^{n+1} t)$$

where u solves the n-th member of the homogeneous Painlevé II hierarchy with boundary condition $u(t) \sim Ai_n(t)$ for $t \to +\infty$.

Remark For $n = 1$ this is the Tracy-Widom formula for the Hastings-McLeod solution of the homogeneous Painlevé II equation [Tracy - Widom, 1994].

About the methods

- [Tracy - Widom, 1994] proved their result through algebraic methods.
- [Ledoussal - Majumdar - Schehr, 2018] extended this method.

On the other side ...

- [Kapaev - Hubert, 1999] proved T - W formula via a Riemann-Hilbert approach. The main idea is to write

$$(x-y)K_{t,n}(x,y) = \sum_{\ell=1}^{n+1} f_\ell(x)g_\ell(y),$$

  and to associate a $(n+1)\times(n+1)$ matrix-valued RH problem (IIKS theory).

- [Cafasso - Claeys - Girotti, 2019] used first Fourier transformations

$$F_n(t) = \det(1-\mathcal{L}_n), \quad \text{with} \quad (\lambda-\mu)L_n(\lambda,\mu) := \sum_{j=1}^{2} u_j(\lambda)v_j(\lambda)$$

  and then associated a 2×2 matrix-valued RH problem for any n.
  Its solution is then used to deduce the Lax pair of the Painlevé II hierarchy.

Outline

1. Introduction
   - The Painlevé II hierarchy
   - Higher order Airy kernels
2. Our object of study
   - Their finite temperature version
   - The analogue Tracy-Widom formula
3. Main ideas for the proof
   - Manipulating the kernels
   - Construction of operator-valued RH problem
   - The derivation of the operator-valued Lax pair
   - Conclusion

The finite-temperature higher order Airy kernels

Definition

Consider now the integral operators $\mathcal{K}^{w}_{t,n}$ acting on $L^2(\mathbb{R}_+)$ with the finite temperature higher order Airy kernel

$$K^{w}_{t,n}(x,y) := \int_{\mathbb{R}} Ai_n(x+z+t)Ai_n(y+z+t)w(z)dz.$$

Remark Consider the weight $w(z)$ as the Fermi factor, i.e. $w(z) = w_\alpha(z) = \dfrac{1}{1+e^{-\alpha z}}$, with $\alpha > 0$.

The aim

To generalize the Tracy-Widom formula for the Fredholm determinant $D_n(t) := \det(1 - \mathcal{K}^{w}_{t,n})$ of the "finite temperature" version of the Airy kernels.

Occurences of $D_n(t)$

* **Determinantal point processes**.
  [Soshnikov, 2000] D_n is (again) the probability distribution of the largest particle of the determinantal point process defined through kernels $K^w_{t,n}$.

* **Fermionic systems at finite temperature**.
  [LeDoussal - Majumdar - Schehr, 2018] With $w = w_\alpha$, D_n describes the same relevant statistical quantity in the model for free fermions in anharmonic traps cited before, but at finite temperature in this case.

* **KPZ equation**.
  [Amir - Corwin - Quastel, 2011] With $w = w_\alpha$, D_1 is used to express the probability distribution of the KPZ solution with narrow wedge initial condition.

* **Random matrices**.
  [Johansson, 2007] With $w = w_\alpha$, D_1 appeared in the grand canonical scaling limits in study of the Moshe-Neuberger-Shapiro model.

Outline

1. Introduction
   - The Painlevé II hierarchy
   - Higher order Airy kernels

2. Our object of study
   - Their finite temperature version
   - The analogue Tracy-Widom formula

3. Main ideas for the proof
   - Manipulating the kernels
   - Construction of operator-valued RH problem
   - The derivation of the operator-valued Lax pair
   - Conclusion

The main statement

Theorem (Bothner - Cafasso - T., 2021)

For every $(t, n) \in \mathbb{R} \times \mathbb{N}$,

$$\frac{d^2}{dt^2} \ln D_n(t) = -\int_{\mathbb{R}} u^2(t|x)w'(x)dx$$

where $u(t|x) \equiv u(t|x; n)$ *solves the n-th member of the integro-differential Painlevé II hierarchy and it is such that*

$$u(t|x) \sim \mathrm{Ai}_n(t+x)$$

as $t \to +\infty$, *pointwise in* $x \in \mathbb{R}$.

- [Amir - Corwin - Quastel, 2011] , [Bothner, 2020], [Cafasso- Claeys - Ruzza, 2021] the result for $n = 1$;
- [Krajenbrink, 2020] the general case through a "Tracy-Widom" approach.

The Painlevé II integro-differential hierarchy

The compatibility condition gives the n-th member of the integro-differential Painlevé II hierarchy

$$\boxed{(t+x)u(t|x) = -\big((\mathcal{L}_+^u\mathcal{L}_-^u)^n u\big)(t|x)}$$

Given a function $\mathbb{R}^2 \ni (t,x) \mapsto f(t|x)$, we define

$$(\mathcal{L}_+^u f)(t|x) := i(D_t f)(t|x) - i\big\langle (D_t^{-1}\{u,f\})(t|x,\cdot), u\big\rangle - 2i\big(D_t^{-1}\langle u,f\rangle\big)u(t|x),$$
$$(\mathcal{L}_-^u f)(t|x) := i(D_t f)(t|x) + i\big\langle (D_t^{-1}[u,f])(t|x,\cdot), u\big\rangle,$$

where

- $[\alpha,\beta] := \alpha\otimes\beta - \beta\otimes\alpha$ is intended as rank two integral operator with kernel

$$[\alpha,\beta](t|x,y) = \alpha(t|x)\beta(t|y) - \beta(t|x)\alpha(t|y),$$

- $\{\alpha,\beta\} := \alpha\otimes\beta + \beta\otimes\alpha$ the same but with kernel

$$\{\alpha,\beta\}(t|x,y) = \alpha(t|x)\beta(t|y) + \beta(t|x)\alpha(t|y),$$

- $\langle\cdot,\cdot\rangle$ indicates

$$\langle f,g\rangle := \int_{\mathbb{R}} f(t|x)g(t|x)w'(x)dx.$$

First equations of the hierarchy

The first members of the hierarchy read as

$$n = 1: \quad (t+x)u = u'' - 2u\langle u, u\rangle,$$

$$\begin{aligned} n = 2: \quad -(t+x)u = {} & u'''' - 4u''\langle u, u\rangle - 8u'\langle u', u\rangle - 6u\langle u, u''\rangle \\ & - 2u\langle u', u'\rangle + 6u\langle u, u\rangle^2, \end{aligned}$$

$$\begin{aligned} n = 3: \quad (t+x)u = {} & u'''''' - 6u''''\langle u, u\rangle - 8u\langle u'''', u\rangle - 24u'''\langle u', u\rangle \\ & - 19u'\langle u, u'''\rangle - 13u\langle u''', u'\rangle - 31u''\langle u'', u\rangle \\ & - 11u\langle u'', u''\rangle - 25u''\langle u', u'\rangle - 45u'\langle u'', u'\rangle \\ & + 15u''\langle u, u\rangle^2 + 55u\langle u, u\rangle\langle u'', u\rangle + 60u'\langle u', u\rangle\langle u, u\rangle \\ & + 25u\langle u', u'\rangle\langle u, u\rangle + 55u\langle u', u\rangle^2 - 20u\langle u, u\rangle^3. \end{aligned}$$

Here $'$ states for the derivation w.r.t. the real parameter t.

Remark With $w'(x) = \delta_0(x)$ these equations coincide with the classical higher order Painlevé II equations for $u(t|0)$.

Outline

1 Introduction
- The Painlevé II hierarchy
- Higher order Airy kernels

2 Our object of study
- Their finite temperature version
- The analogue Tracy-Widom formula

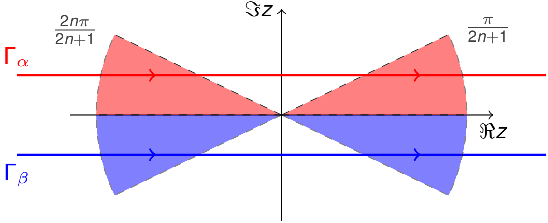
3 Main ideas for the proof
- Manipulating the kernels
- Construction of operator-valued RH problem
- The derivation of the operator-valued Lax pair
- Conclusion

Integral representations of the higher order Airy functions

$Ai_n(x)$ admits the contour integral representations

$$Ai_n(x) = \frac{1}{2\pi}\int_{\Gamma_\alpha} e^{i\psi_n(\alpha,x)}d\alpha = \frac{1}{2\pi}\int_{\Gamma_\beta} e^{-i\psi_n(\beta,x)d\beta},$$

with $\psi_n(\lambda, x) := \frac{\lambda^{2n+1}}{2n+1} + x\lambda$ for $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$.



The relevant *integrable* operator

Let fix the contour $\Sigma := \Gamma_\beta \cup \mathbb{R} \cup \Gamma_\alpha$ where $\Gamma_\beta = \mathbb{R} - i\Delta$, and Γ_α is its reflection across $\mathbb{R}$.

We can express the Fredholm determinant $D_n(t)$ as

$$\boxed{D_n(t) = \det(1 - C_{t,n})}$$

where $C_{t,n}$ is an integral operator on $L^2(\Sigma)$ with kernel

$$(\xi - \eta)C_{t,n}(\xi, \eta) = \int_{\mathbb{R}} \left(k_1(\xi|z)m_1(\eta|z) + k_2(\xi|z)m_2(\eta|z)\right) w'(z)dz,$$

where

$$k_1(\zeta|y) := \frac{1}{2\pi} e^{\frac{i}{2}\psi_n(\zeta,2t+2y)}\chi_{\Gamma_\alpha}(\zeta), \quad k_2(\zeta|y) := \frac{1}{2\pi} e^{-\frac{i}{2}\psi_n(\zeta,0)}\chi_{\Gamma_\beta}(\zeta),$$
$$m_1(\zeta|x) := e^{-\frac{i}{2}\psi_n(\zeta,2t+2x)}\chi_{\Gamma_\beta}(\zeta), \quad m_2(\zeta|x) := e^{\frac{i}{2}\psi_n(\zeta,0)}\chi_{\Gamma_\alpha}(\zeta).$$

Remark To the "∞-dimensional integrable operator" $C_{t,n}$ we can associate an *operator valued* Riemann-Hilbert problem.

Outline

1. Introduction
   - The Painlevé II hierarchy
   - Higher order Airy kernels
2. Our object of study
   - Their finite temperature version
   - The analogue Tracy-Widom formula
3. Main ideas for the proof
   - Manipulating the kernels
   - Construction of operator-valued RH problem
   - The derivation of the operator-valued Lax pair
   - Conclusion

Some definitions

Notation

- $\mathcal{H}_p := \bigoplus_{j=1}^{p} L^2(\mathbb{R}, w'(z)dz)$.
- $\mathcal{I}(\mathcal{H}_p)$ the space of Hilbert-Schmidt integral operators on $\mathcal{H}_p$ with kernel in $L^2(\mathbb{R}^2, w'(z)dz \otimes w'(z)dz; \mathbb{C}^{p\times p})$.

Definition

$(M_i(\zeta) \otimes K_j(\zeta)) \in \mathcal{I}(\mathcal{H}_1)$ a $\Sigma \ni \zeta$-family of rank 1 integral operators with kernels

$$(M_i(\zeta) \otimes K_j(\zeta))(x, y) = m_i(\zeta|x)k_j(\zeta|y).$$

Remark Since $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ we have

$$(M_1(\zeta) \otimes K_1(\zeta))(x, y) = 0 = (M_2(\zeta) \otimes K_2(\zeta))(x, y)$$

The operator-valued RH problem

Given $(t, n) \in \mathbb{R} \times \mathbb{N}$, determine $\mathbf{X}(\zeta) = \mathbb{I}_2 + \mathbf{X_0}(\zeta; t, n)$, such that

(1) $\mathbf{X}_0(\zeta) \in \mathcal{I}(\mathcal{H}_2)$ with kernel $\mathbf{X}_0(\zeta|x, y)$ analytic in $\mathbb{C} \setminus \Sigma$.

(2) $\mathbf{X}(\zeta)$ admits continuous boundary values $\mathbf{X}_\pm(\zeta) \in \mathcal{I}(\mathcal{H}_2)$ on Σ, which satisfy $\mathbf{X}_+(\zeta) = \mathbf{X}_-(\zeta)\mathbf{G}(\zeta)$ with

$$\mathbf{G}(\zeta) = \mathbb{I}_2 + 2\pi i \lambda^{\frac{1}{2}} \begin{bmatrix} M_1(\zeta) \otimes K_1(\zeta) & M_1(\zeta) \otimes K_2(\zeta) \\ M_2(\zeta) \otimes K_1(\zeta) & M_2(\zeta) \otimes K_2(\zeta) \end{bmatrix}, \quad \zeta \in \Sigma.$$

(3) $\mathbf{X}(\zeta) \sim \mathbb{I}_2$ for $|\zeta| \to \infty$ with a particular condition on the operator norm of $\mathbf{X}_0$.

Solution of the RH problem

1. The RH problem is uniquely solvable.
2. For every $(t, n) \in \mathbb{R} \times \mathbb{N}$, the solution $\mathbf{X}(\zeta)$ admits an integral representation

$$\mathbf{X}(\zeta) = \mathbb{I}_2 + \int_\Sigma \begin{bmatrix} N_1(\eta) \otimes K_1(\eta) & N_1(\eta) \otimes K_2(\eta) \\ N_2(\eta) \otimes K_1(\eta) & N_2(\eta) \otimes K_2(\eta) \end{bmatrix} \frac{d\eta}{\eta - \zeta}, \quad \zeta \in \mathbb{C} \setminus \Sigma.$$

   Here $N_i(\eta)$ are the operators on $\mathcal{H}_1$ which multiply by the functions $n_i(\eta|x)$ determined via

$$\left(I - C^*_{t,n} \restriction_{L^2(\Sigma)}\right) n_i(\cdot|x) = m_i(\cdot|x), \quad i = 1, 2.$$

   Moreover, $\mathbf{X}(\zeta)$ is invertible and its inverse admits analogue integral representation.

Remark For $\zeta \in \mathbb{C} \setminus \Sigma$, the asymptotic representation of $\mathbf{X}(\zeta)$

$$\mathbf{X}(\zeta) = \mathbb{I}_2 - \frac{\mathbf{X}_1}{\zeta^1} + \mathcal{O}(\zeta^{-2}), \text{ with } X_1^{ml} = \int_\Sigma N_m(\eta) \otimes K_l(\eta) d\eta$$

is used to define $U := X_1^{12}$ and $V := X_1^{21}$, with kernels $U(x, y) = V(y, x)$ and such that they are y-independent.

The next steps

Solution $\mathbf{X}(\zeta) = \mathbf{X}(\zeta; t, n)$ of the operator-valued RH problem

$\swarrow \qquad \searrow$

$\mathbf{N}(\zeta) = \mathbf{X}(\zeta)\mathbf{M}(\zeta)$ for

$$\mathbf{N}(\zeta) := \begin{bmatrix} N_1(\zeta) \\ N_2(\zeta) \end{bmatrix} \quad \mathbf{M}(\zeta) := \begin{bmatrix} M_1(\zeta) \\ M_2(\zeta) \end{bmatrix}$$

$\downarrow$

$$\frac{\partial \mathbf{N}}{\partial \zeta}(\zeta) = \mathbf{A}(\zeta)\mathbf{N}(\zeta), \quad \frac{\partial \mathbf{N}}{\partial t}(\zeta) = \mathbf{B}(\zeta)\mathbf{N}(\zeta)$$

$\downarrow$

the compatibility condition gives a coupled system of ODEs for U, V

$\downarrow$

n-th integro-differential PII equation for their kernels

$$D_n(t) = \det(1 - C_{t,n})$$

$\downarrow$

$C_{t,n}$ determines the solution $\mathbf{X}(\zeta)$

$\downarrow$

$$\frac{d^2}{dt^2} \ln D_n(t) = - \operatorname*{tr}_{\mathcal{H}_1}(UV)$$

The next steps

Solution $\mathbf{X}(\zeta) = \mathbf{X}(\zeta; t, n)$ of the operator-valued RH problem

$\swarrow \qquad \searrow$

$\mathbf{N}(\zeta) = \mathbf{X}(\zeta)\mathbf{M}(\zeta)$ for

$$\mathbf{N}(\zeta) := \begin{bmatrix} N_1(\zeta) \\ N_2(\zeta) \end{bmatrix} \quad \mathbf{M}(\zeta) := \begin{bmatrix} M_1(\zeta) \\ M_2(\zeta) \end{bmatrix}$$

$\downarrow$

$$\frac{\partial \mathbf{N}}{\partial \zeta}(\zeta) = \mathbf{A}(\zeta)\mathbf{N}(\zeta), \quad \frac{\partial \mathbf{N}}{\partial t}(\zeta) = \mathbf{B}(\zeta)\mathbf{N}(\zeta)$$

$\downarrow$

the compatibility condition gives a coupled system of ODEs for U, V

$\downarrow$

n-th integro-differential PII equation for their kernels

$$D_n(t) = \det(1 - C_{t,n})$$

$\downarrow$

$C_{t,n}$ determines the solution $\mathbf{X}(\zeta)$

$\downarrow$

$$\frac{d^2}{dt^2} \ln D_n(t) = - \operatorname*{tr}_{\mathcal{H}_1}(UV)$$

The next steps

Solution $\mathbf{X}(\zeta) = \mathbf{X}(\zeta; t, n)$ of the operator-valued RH problem

$\swarrow \qquad \searrow$

$\mathbf{N}(\zeta) = \mathbf{X}(\zeta)\mathbf{M}(\zeta)$ for

$$\mathbf{N}(\zeta) := \begin{bmatrix} N_1(\zeta) \\ N_2(\zeta) \end{bmatrix} \quad \mathbf{M}(\zeta) := \begin{bmatrix} M_1(\zeta) \\ M_2(\zeta) \end{bmatrix}$$

$\downarrow$

$$\frac{\partial \mathbf{N}}{\partial \zeta}(\zeta) = \mathbf{A}(\zeta)\mathbf{N}(\zeta), \quad \frac{\partial \mathbf{N}}{\partial t}(\zeta) = \mathbf{B}(\zeta)\mathbf{N}(\zeta)$$

$\downarrow$

the compatibility condition gives a coupled system of ODEs for U, V

$\downarrow$

n-th integro-differential PII equation for their kernels

$$D_n(t) = \det(1 - C_{t,n})$$

$\downarrow$

$C_{t,n}$ determines the solution $\mathbf{X}(\zeta)$

$\downarrow$

$$\frac{d^2}{dt^2} \ln D_n(t) = - \operatorname*{tr}_{\mathcal{H}_1}(UV)$$

Outline

1 Introduction
- The Painlevé II hierarchy
- Higher order Airy kernels

2 Our object of study
- Their finite temperature version
- The analogue Tracy-Widom formula

3 Main ideas for the proof
- Manipulating the kernels
- Construction of operator-valued RH problem
- The derivation of the operator-valued Lax pair
- Conclusion

The operator-valued Lax pair

$\mathbf{N}(\zeta)$ solves the differential equations

$$\frac{\partial \mathbf{N}}{\partial \zeta}(\zeta) = \mathbf{A}(\zeta)\mathbf{N}(\zeta), \quad \frac{\partial \mathbf{N}}{\partial t}(\zeta) = \mathbf{B}(\zeta)\mathbf{N}(\zeta),$$

with $\mathbf{B}(\zeta), \mathbf{A}(\zeta)$ that are polynomials in ζ of form

$$\mathbf{B}(\zeta) = \zeta \mathbf{B}_0 + \mathbf{B}_1, \quad \text{and} \quad \mathbf{A}(\zeta) = \zeta^{2n}\mathbf{A}_0 + \sum_{k=1}^{2n} \mathbf{A}_k \zeta^{2n-k} + \widehat{\mathbf{A}}_{2n}.$$

Some kernels of the operator-valued coefficients are explicitly written

$$\mathbf{B}_0(x,y) := \delta(x-y)\begin{bmatrix} -i & 0 \\ 0 & 0 \end{bmatrix}(w'(y))^{-1}, \mathbf{B}_1(x,y) := \begin{bmatrix} 0 & -iU(x,y) \\ iV(x,y) & 0 \end{bmatrix}$$

$$\mathbf{A}_0(x,y) := \delta(x-y)\frac{1}{2}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}(w'(y))^{-1},$$

$$\widehat{\mathbf{A}}_{2n}(x,y) := \delta(x-y)\begin{bmatrix} -i(t+x) & 0 \\ 0 & 0 \end{bmatrix}(w'(y))^{-1}.$$

Exploiting the compatibility condition

The compatibility condition

$$\mathbf{A}(\zeta)\mathbf{B}(\zeta) - \mathbf{B}(\zeta)\mathbf{A}(\zeta) = \frac{\partial \mathbf{B}}{\partial \zeta}(\zeta) - \frac{\partial \mathbf{A}}{\partial t}(\zeta), \quad \zeta \in \mathbb{C},$$

$$\downarrow$$

using the polynomiality of $\mathbf{B}(\zeta), \mathbf{A}(\zeta)$ in ζ

$$\downarrow$$

is equivalent to

- a system of differential-difference equations that determines all the entries A_k^{ij} of the coefficients $\mathbf{A}_k$ for $k = 1, \ldots, 2n$ in terms of U, V and their t-derivatives,
- a coupled system of ODEs involving U and V.

$$\downarrow$$

Defining $u(t|x) := U(x,x) = U(x,y) = V(y,x) = V(x,x)$, the compatibility condition at the level of the kernels of these operators, coincides exactly with the n-th member of the integro-differential PII hierarchy.

Outline

1. Introduction
   - The Painlevé II hierarchy
   - Higher order Airy kernels
2. Our object of study
   - Their finite temperature version
   - The analogue Tracy-Widom formula
3. Main ideas for the proof
   - Manipulating the kernels
   - Construction of operator-valued RH problem
   - The derivation of the operator-valued Lax pair
   - Conclusion

Formula for the logarithmic derivative of the Fredholm determinant

Lemma

We have that $\frac{d^2}{dt^2}\ln D_n(t) = -\int_{\mathbb{R}} u^2(t|x)w'(x)dx.$

Indeed, we first compute

$$\frac{d}{dt}\ln \underbrace{D_n(t)}_{\det(1-C_{t,n})} = -\operatorname*{tr}_{L^2(\Sigma)}\left((1-C_{t,n})^{-1}\frac{\partial}{\partial t}C_{t,n}\right) = -i\operatorname*{tr}_{\mathcal{H}_1}\int_{\Sigma} N_1(\xi)\otimes K_1(\xi)\,d\xi$$

and then

$$\begin{aligned}\frac{d^2}{dt^2}\ln D_n(t) &= -\operatorname*{tr}_{\mathcal{H}_1}(UV) = -\int_{\mathbb{R}}\int_{\mathbb{R}} U(x,y)V(y,x)\,w'(y)dy\,w'(x)dx\\ &= -\int_{\mathbb{R}} u^2(t|x)w'(x)dx.\end{aligned}$$

Where $u(t|x)$

- solves the n-th member of the integro-differential PII hierarchy;
- for $t\to+\infty$ it behaves like

$$u(t|x) = \int_{\Sigma} n_1(\eta|x)k_2(\eta|x)d\eta \sim \int_{\Sigma} m_1(\eta|x)k_2(\eta|x)d\eta = Ai_n(t+x).$$

Future directions

- This operator-valued Riemann-Hilbert approach can be used to study other integral operators of the same type of $C_{t,n}$ and to derive connections with *new* integrable systems.
- In the classical (zero temperature) case, the Riemann-Hilbert approach allows to compute, via the non linear steepest descent method, the *large gap asymptotics* of the relevant Fredholm determinants (Cafasso - Claeys - Girotti, 2019). What about *large gap asymptotics* of $D_n(t)$ using o.-v. RH problems?

Remark [Its - Kozlowski, 2014] study of the asymptotic behavior of the generalized shifted sine kernel throgh o.-v. RH problems.

Thank you!

The work of Its-Kozlowski on the c-shifted generalized sine kernel

[Its - Kozlowski, 2014] used operator-valued Riemann-Hilbert problems to study the large x - asymptotic behavior of the Fredholm determinant of an integral operator $\mathcal{V}$ on $L^2([a,b])$ of *c-shifted type*, with kernel

$$V(\lambda,\mu) := \frac{icF(\lambda)}{2\pi i(\lambda-\mu)}\left(\frac{e^{\frac{ix}{2}(p(\lambda)-p(\mu))}}{(\lambda-\mu)+ic} + \frac{e^{\frac{ix}{2}(p(\mu)-p(\lambda))}}{(\lambda-\mu)-ic}\right)$$

where F, p are functions satisfying some technical assumptions.

Theorem (Its - Kozlowski, 2014)

Let $\mathcal{V}_0$ be the integral operator on $L^2([a,b])$ with kernel $V_0(\lambda,\mu) = \frac{F(\lambda)}{\pi(\lambda-\mu)}\sin\left(\frac{x}{2}(p(\lambda)-p(\mu))\right)$. Then one has the large x-asymptotic behavior

$$\frac{\det(1+\mathcal{V})}{\det(1+\mathcal{V}_0)} = \det(1+\mathcal{U}_+)\det(1+\mathcal{U}_-)(1+o(1)),$$

where $\mathcal{U}_\pm$ are integral operators on $L^2(C\,[a,b])$ with kernels independent on x.

And the large x-asymptotic behavior of $\det(1+\mathcal{V}_0)$ is known from [Kitanine - Kozlowski - Maillet - Slavnov - Terras, 2009] .

Their main ideas

(1) Introduction of a parameter ℓ, to consider a family of kernels

$$V_\ell(\lambda,\mu) := \frac{icF(\lambda)}{2\pi i(\lambda-\mu)}\left(\frac{e^{\frac{ix}{2}(p(\lambda)-p(\mu))}}{\ell(\lambda-\mu)+ic}+\frac{e^{\frac{ix}{2}(p(\mu)-p(\lambda))}}{\ell(\lambda-\mu)-ic}\right)$$

so that $V_{t=1}(\lambda,\mu) = V(\lambda,\mu)$ and $V_{t=0}(\lambda,\mu)$ is the generalized sine kernel.

(2) The kernel can be represented in form

$$V_\ell(\lambda,\mu) = \frac{-F(\lambda)}{2\pi i(\lambda-\mu)}\int_0^{+\infty}\left(k_1(\lambda;\ell,s)m_1(\mu;\ell,s)+k_2(\lambda;\ell,s)m_2(\mu;\ell,s)\right)ds$$

thus an operator-valued RH problem on $\mathbb{R}_+$ is associated in the canonical way.

(3) Non-linear steepest descent method is applied to this RH problem, generalizing the steps done in the study of the case $\ell = 0$ (on standard matrix-valued RH problems).

(4) $\partial_\ell \ln(\det(1+\mathcal{V}_\ell))$ is computed in terms of the solution of the o.-v. RH problem and in the large x limit its integration between 0 and 1 gives the result.

Could α^{-1} play the same role of the parameter ℓ in the Airy case?

Given the work of [Its - Kozlowski, 2014] and given the results on the large gap asymptotics in [Cafasso - Claeys - Girotti, 2019] for the zero-temperature case

$$F_n(t) = \det(1 - K_{n,t})$$

the naive question would be: could we deduce a similar result, by taking ℓ here as the "inverse temperature", for

$$\partial_\ell \det(1 - K_{t,n}^{w_\ell})$$

so that integrating between $\ell = 0$ to $\ell = \alpha$ finite, in the regime $t \to -\infty$, something could be said about

$$\frac{D_n(t)}{F_n(t)} := \frac{\det(1 - K_{t,n}^{w_\alpha})}{\det(1 - K_{t,n})} = \quad \ldots\ldots \quad ?$$