Cluster algebras and monodromy manifolds of irregular type (séminaire GA, Angers, 9/12/21)

Thm (Bertola–T. 2021, ArXiv 2104.13784)

1) The monodromy manifold $\mathcal{G}_K$ associated to a linear system of ODE (in $\mathbb{C}$) of type
$$\frac{d\varphi(\lambda)}{d\lambda} = A(\lambda)\varphi(\lambda) \quad , \text{ with } A(\lambda) = \sum_{j=0}^{k} A_j \lambda^j \, , A_j \in sl_2$$
is a symplectic manifold with two-form given by
$$\mathcal{W}_K = \frac{1}{2}\sum_{\ell=1}^{2k+3} \mathrm{Tr}\left(He^{-1} dHe \wedge S_e^{-1} dS_e \right)$$
, S_e being the Stokes matrices, $\ell = 1, .., 2k+2$
$He = S_1 \cdots S_e$, $S_{2k+3} = \lambda^{\sigma_3}$.

2) Moreover, there exist coordinates $\{y_j\}_{j=1}^{2k = \dim \mathcal{G}_K}$ parametrizing $\mathcal{G}_K$ that are log-canonical for $\mathcal{W}_K$, i.e. the induced Poisson structure in those coordinates is written as:
$$\{y_i, y_j\} = (P_k)_{ij}\, y_i y_j \qquad \forall i,j = 1, .., 2k$$

3) Up to a constant integer factor, this is the compatible Poisson structure defined on the cluster manifold associated to the $\mathcal{X}$-cluster algebra of type A_{2k} with one frozen variables.

4) Finally, the Poisson brackets are the "linearization" of the classical Flaschka-Newell P.b. computed by the authors in the '80, for the monodromy parameters.

Motivations

~ ~ ~

1) Why is it interesting to study monodromy manifolds?
From the "integrable systems" point of view, the interest is given from the connection existing among isomonodromic deformations theory and Painlevé equations.
In particular: every P.eq. has a representation with a Lax pair
$$\begin{cases} \dfrac{d\varphi}{d\lambda} = A(\lambda,t)\varphi \\ \dfrac{d\varphi}{dt} = B(\lambda,t)\varphi \end{cases}$$
that describes the isomonodromic deformations of the first eq. (i.e. all the coefficients $A(\lambda,t)$ s.t. the set of essential monodromy data of the first eq. is left invariant)
In particular, this implies that ~~for~~ to every point of the (correspondent) monodromy manifold corresponds a certain solution of the (correspondent) Painlevé eq.
(the "Riemann-Hilbert" correspondence, see ~~Kapa~~ -Its-Fokas-Novokshenov, Yu).
Thus understanding geometrical properties of monodromy manifolds can be used in this sense.
Point problematic: why for systems having only regular singularities, monodromy manifolds are determined as linear representation of the fundamental group of appropriately punctured Riemann sphere, thus its geometry is encoded by character varieties.

E.g. PVI ⟷ $SL_2(\mathbb{C})$ -character variety of $\mathbb{CP}^1 \setminus \{4 \text{ holes}\}$.
But all the other Painlevé equations are encoded by Lax pairs of system with irregular singularities, thus involving Stokes phenomena and thus having monodromy manifolds that no longer can be described through charact. varieties.

② Why the 2-form ω_K? (counterpart)
This 2-form is a sort of generalization of the Goldmann Poisson brackets classically defined for character varieties of Riemann surfaces (so the classical structure that is defined on the regular singular case).

③ Why the link with cluster algebras?
For general character varieties, the connection with cluster algebras and their usage to express symplectic/Poisson structure in log-canonical form was largely studied by Fock-Goncharov (2006).
Recently, their formalism has been applied, with the same aim, to expresse the extension of the Goldman P. structure of arbitrary punctured R. surfaces (Bertola-Korotkin, 2019).
Thus the connection with cluster algebras in the study of this "irregular" monodromy manifold is not surprising, but also important since it adds another example of symplectic/Poisson manifold to the list of those manifolds having a compatible P. structure with the cluster algebra structure (which started long ago with the real Grassmannian, for more info see Gekhtman-Shapiro-Vainshtein "Cluster Algebras and Poisson geometry", Ch 4.2).

④ What about F-N Poisson structure?
This F-N P. str. was the first example of Poisson structure computed on monodromy parameters for an irregular singular case. Finding back this P. str. through the linearization in terms of cluster variables allows us to give new insight on that work of the '80, forgotten for a long time.

~ ~ ~

Method

Part 1

$\boxed{\text{"Stokes" manifold } \mathcal{G}_K}$

its construction is induced from the theory of solutions of linear ODE's in $\mathbb{C}$

Part 2

$\boxed{\text{"Cluster algebras of } A_{2K} \text{ type"}}$

For a given Dynkin diagram of type A_{2K} there is associated a kind of cluster algebra/manifold/structure

The link between the 2 is given from the study of the symplectic structure on $\mathcal{G}_K$ and in particular by using the theory of the "standard 2-form" associated to a graph with connection (↳ some surface).

Part 1 "Stokes manifold"

We study the monodromy manifold associated to the linear ODE

$$(*) \quad \frac{d\varphi(z)}{dz} = A(z)\varphi(z) \qquad \underset{A_k \neq 0}{A(z) = \sum_{j=1}^{k} A_j z^j}$$

, i.e. having only 1 irregular singular point at ∞ of Poincaré rank $k+1$ (consider the differential $A(z)dz$ on $\mathbb{CP}^1$)

with the following assumptions:

1. $A_j \in sl_2(\mathbb{C})$ w.l.g.
2. A_k has simple eigenvalues (This is important and non neglectable condition, since it would modify the form of the formal solution of $(*)$ near ∞, and thus the monodromy data)

The local theory of solutions for systems like $(*)$ says that:

- .) around z_0 non singular point of $A(z)dz$: $\exists!$ sol. of $(*)$ holomorphic in D_{z_0} for any fixed initial condition $\varphi(z_0) = \varphi_0 \in GL_2(\mathbb{C})$.
- ..) around z_0 a singular regular point (pole of order 1) of $A(z)dz$, then the sol. of $(*)$ in $D_{z_0} \setminus \{z_0\}$ exist, is holomorphic, and its given by

$$\varphi(z) = \underset{\text{holomorphic}}{\hat{\varphi}(z)}\, \underbrace{(z-z_0)}_{\text{or } \frac{1}{z} \;=\zeta}{}^{T_0} \qquad \text{with } T_0 \text{ s.t. } A(z)dz = \sum_{k \geq -1} M_{k+1}\zeta\, d\zeta$$

and $M_0 = P T_0 P^{-1}$, (T_0 diagonal)

and $\hat{\varphi}(z_0) = P$.

- ...) What does it happen when we have an irregular singular point, as in this case?

There exist a formal solution of $(*)$ written as:

$$\varphi_{form}(z) = \hat{Y}(z)\, z^{-L} e^{T(z)} \qquad \text{where } \hat{Y}(z) = G_0\left(\mathbb{1}_2 + \sum_{j \geq 1} \frac{Y_j}{z^j}\right) \in SL_2(z^{-1})$$

where:

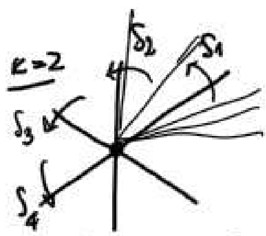
- $\rightarrow$ G_0 is the diagonalizing matrix for A_k: $A_k = G_0 T_{k+1} G_0^{-1}$
- $\rightarrow$ L, T are diagonal trace-less matrices, in particular:

$$L = \begin{pmatrix} l & 0 \\ 0 & -l \end{pmatrix}, \quad T(z) = T_{k+1}\frac{z^{k+1}}{k+1} + \dots + T_1 z, \quad T_j \in \text{Cartan subalgebra of } sl_n \text{ (diagonal traceless)}$$

But this is just a formal solution, since it does not converges at ∞. Here there is the Stokes phenomenon going on, i.e.:

the plan can be partitioned in $2\cdot(k+1) = 2k+2$ overlapping sectors S_μ of equal angular width $\pi/_{(k+1)+\varepsilon}$, arranged in counter clockwise order and s.t. within each sector $\exists!$ analytic (fund.) sol. φ_μ of $(*)$ which has asymptotic behavior:

$$\varphi_\mu \longrightarrow \varphi_{form} \quad \text{for } |z| \to \infty,\ \arg z \in S_\mu$$



For each one of these $2k+2$ overlapping sectors, one can define the following matrices, called Stokes matrices, that describe the passage from one unique solution to the other:

the Stokes graph

$$S_\mu = \Psi_\mu^{-1}(z)\,\Psi_{\mu+1}(z) \qquad z \in S_\mu \cap S_{\mu+1}. \text{ (they are constant in } z)$$

Moreover: Taking $t_1,\dots,t_n$ entries of T_{k+1} in increasing order of $\mathbb{R}(t_j e^{\theta_0})$ then, all the S_μ are triangulars with unit diagonal (and they alternate upper/lower triangularity).

Finally, the monodromy data are given by the collection of the $2k+2$ Stokes matrices together with the formal monodromy exponent L. However, they are not independent (because there are no other singular points of (*), the solutions ψ_n extends uniquely to entire matrix-valued functions):

$$S_1 \cdot S_2 \cdot \ldots \cdot S_{2k+2}\, e^{2\pi i L} = \mathbb{1}_2 .$$

Thus in the end we define the "Stokes manifold" as the following algebraic variety:

Def $\mathcal{G}_k := \left\{ S_1,\dots, S_{2k+2}, L \text{ s.t. } \overset{\begin{pmatrix}1&s_1\\0&1\end{pmatrix}}{S_1}\cdot\overset{\begin{pmatrix}1&0\\s_2&1\end{pmatrix}}{S_2}\cdots \overset{\begin{pmatrix}1&0\\s_{2k+2}&1\end{pmatrix}}{S_{2k+2}}\,\lambda^{-\sigma_3} = \mathbb{1}_2 \right\}$ $(\lambda = e^{2\pi i l})$

Notice that $\dim \mathcal{G}_k = 2k$ (even: good for being symplectic).

Rmk P. Boalch "Quasi-Hamiltonian geometry of meromorphic connections" proved that all these types of manifolds arising from monodromy theory are symplectic in a very abstract and general way. Here our approach is more practical and specialized "ad hoc" for this case of monodromy manifold, that in any case should fit in the general picture of "wild character varieties" introduced by P. Boalch.

Def On $\mathcal{G}_k$ we consider the 2-form

$$\omega_k = \frac{1}{2}\sum_{l=1}^{2k+3} \mathrm{Tr}\left(H_l^{-1}\, dH_l \wedge S_l^{-1}\, dS_l\right)$$

Rmk: there is an abstract/general way to prove that ω_k is symplectic, and the idea is as follows:

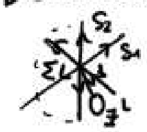
→ consider, on the side of rational matrix coefficients, the "universal symplectic structure" of Krichever-Phong defined on the symplectic leaves of the Lie-Poisson structure defined for the matrix coeff. entries. It reads as

$$\omega_{KK} = -\operatorname*{res}_{z=\infty} \mathrm{Tr}\left(A(z)\, \delta G(z)\, G(z)^{-1} \wedge \delta G(z)\, G(z)^{-1}\right) dz$$

and it has symplectic potential (here $G(z)^{-1} A(z)\, G(z) = \underset{\text{diagonal}}{D(z)}$)

$$\Theta = \operatorname*{res}_{z=\infty} \mathrm{Tr}\left(D(z)\, G(z)^{-1}\, \delta G(z)\right) \quad \text{i.e. } \delta\Theta = \omega_{KK}.$$

→ consider, on the sides of monodromy manifold, the RH pb given by the Stokes graph



$\Psi_+(z) = \Psi_-(z) J(z)$ with $J(z) = S_e$ or z^L.

and the Malgrange form associated to this RHpb:

$$\Theta_M := \int_\Sigma \mathrm{Tr}\left(\Psi_-^{-1}(z,s)\, \Psi_-'(z,s)\, \delta J(z,s)\, J^{-1}(z,s)\right) \frac{dz}{2\pi i} .$$

Then from (Bertola, 2011) follows that:

$$\delta\Theta_M = -\frac{1}{2}\int_\Sigma \mathrm{Tr}\Big(\underset{\underset{\delta J \cdot J^{-1}}{\shortparallel}}{\Xi}{}'(z) \wedge \Xi(z)\, \frac{dz}{2\pi i}\Big) - \frac{1}{4\pi i} \sum_{v \in V(\Sigma)} \sum_{\ell=1}^{n_v} \mathrm{Tr}\left(H_\ell^{-1}(v)\, \delta H_\ell(v) \wedge J_\ell^{-1}(v)\, \delta J_\ell(v)\right)$$

→ prove that $\vartheta = \Theta_M + \underbrace{\delta \mathrm{Tr}(\dots)}_{\text{exact differential}}$ by using the monodromy map

and the fact that Ψ the sol. of the RHpb is given by the sol. of the ODE (*) in a proportional way. Conclude that up to $2\pi i$ $\delta\theta = \omega_K$ i.e. the pullback of ω_K through the monodromy map coincide with the LP struct. on sympl. leaves.

Part 2 "Cluster algebras and quivers"

Consider a quiver Q, i.e. an oriented graph without loops, with vertices q_ℓ, $\ell = 1 \dots m$.
Consider variables K_ℓ associated to each vertex of Q.

<u>Def</u> The adjacency matrix of Q is the skew-symmetric, integer-valued matrix

$$B_{k,\ell} = \#\{q_k \to q_\ell\} - \#\{q_\ell \to q_k\} \quad , k,\ell = 1 \dots m.$$

<u>Def</u> $\{k_\ell\}_{\ell=1}^m$ is the initial seed associated to the quiver Q.

<u>E.g.</u>

$$k_1 \leftarrow k_2 \leftarrow k_3 \leftarrow k_4$$

<u>Def</u> A mutation of Q w.r.t. the vertex q_k (K_k variable) is a new quiver $\mu_k(Q)$, with the same set of vertices, but new edges defined as follows

1) for every $q_i \to q_k \to q_\ell$ add $q_i \to q_\ell$
2) reverse every edge with extreme q_k
3) remove loops if any

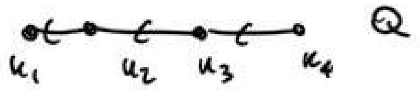
<u>Rmk</u>: Equivalently one can say that $\mu_k(Q)$ has adjacency matrix obtained from the previous one B, through the following transf.

$$\mu_k(B)_{st} = \begin{cases} -B_{st} & \text{for } s = k,\ k = t \\ B_{st} + \mathrm{sign}(B_{sk}) \underbrace{[B_{sk}, B_{kt}]_+}_{\text{denotes the positive part}} & \text{otherwise} \end{cases}$$

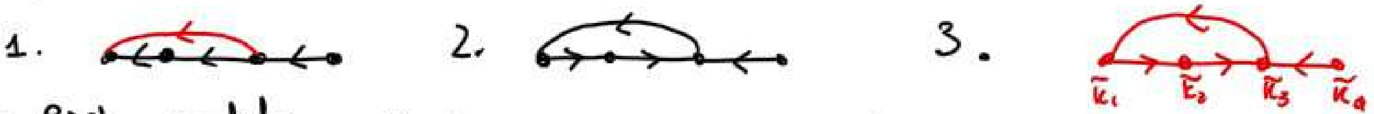
0 if <0, itself if >0,

Rmk It's an involution

E.g.



$\mu_2(Q)$:



To each mutation of the quiver is associated a new set of variables $\{\tilde{x}_\ell\}_{\ell=1}^m$ obtained from the previous ones through birational maps called $\mathcal{X}$-cluster mutation relations.

Def To the mutated quiver $\mu_k(Q)$, one associates to each vertex a new variable $\{\tilde{x}_\ell\}_{\ell=1}^m$, obtained from $\{x_\ell\}_{\ell=1}^m$ as follows:

$$\mu_k(x_\ell) = \begin{cases} \dfrac{1}{x_k} & \text{if } \ell = k \\ x_\ell \cdot \dfrac{x_k^{[B_{\ell k}]_+}}{(1+x_k)^{B_{\ell k}}} & \text{if } \ell \neq k \end{cases}$$

Rmk1 Again it's an involution.

Rmk2: essentially the only variables that changed are x_k itself and the one adjacent to x_k. Moreover for x_ℓ adjacent to x_k

$\mu_k(x_\ell) = \dfrac{x_\ell x_k}{(1+x_k)}$ if $q_\ell \to q_k$, $\mu_k(x_\ell) = x_\ell(1+x_k)$ if $q_k \to q_\ell$

E.g. In the previous mutation we have:

$$\tilde{x}_2 = \frac{1}{x_2}, \quad \tilde{x}_4 = x_4, \quad \tilde{x}_1 = x_1(1+x_2), \quad \tilde{x}_3 = x_3 \frac{x_2}{(1+x_2)}.$$

Def Given the pair $(Q, \vec{x})$, the $\mathcal{X}$-cluster algebra $\mathcal{A}_{\mathcal{X}}(Q_x)$ is the subring of all polynomials in $\{x_\ell\}_{\ell=1}^m$ and its seeds obtained by subsequent mutations

(meaning the seeds that are mutation equivalent to $(Q, \vec{x})$ i.e. for which exists a sequence of mutation to pass from one to the other).

Rmk (Thm 3.26 G-S-V) For B sign-skew-symm: all the generated cluster algebras are of finite type. (i.e. there exist a finite number of nonequivalent seeds).

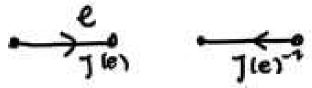
Def For a given cluster algebra, one can define the associated Cluster manifold as $\mathcal{X} := \mathrm{Spect}_{\mathrm{smooth}}(\mathcal{A}(Q))$ (meaning that one takes by hands the smooth part of this object).

Rmk On this rather complicated object, one can define a Poisson/symplectic (eventually) structure, as: $\{ \kappa_i, \kappa_j \} = \underbrace{B}_{\text{the adjacency matrix of the quiver.}} \kappa_i \kappa_j$

Part 3 "The ideas behind the proof of the theorem"

In order to prove that the Stokes manifold $\mathcal{G}_k$ can be parametrized through cluster variables $\{y_j\}_{j=1}^{2k}$ that are log canonical for ω_k, the main tool we need to use is: the theory of standard 2-form associated to a graph with connection.

<u>Def</u> Let Σ be an oriented graph $\hookrightarrow$ surface (for us $\mathbb{CP}^1$) with connection J, i.e. a map $J : E(\Sigma) \to SL_2(\mathbb{C})$ s.t.:
$$e \mapsto J(e)$$

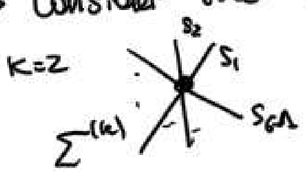
1. $J(-e) = J(e)^{-1}$ 
2. $\forall v \in V(\Sigma)$, n_v being the valence of v, then: $J(e_1) \cdot \ldots \cdot J(e_{n_v}) = \mathbb{1}_2$ 
where each edge e_i is oriented away from v and they are counted counterclockwise starting from any of them.

<u>Def</u> the 2-form associated to (Σ, J) is canonically defined as:
$$\Omega(\Sigma) = \sum_{v \in V(\Sigma)} \sum_{k=1}^{n_v - 1} \mathrm{Tr}\left(\left(H^{(v)}_{[1:k]}\right)^{-1} d H^{(v)}_{[1:k]} \wedge \left(J_k^{(v)}\right)^{-1} dJ_k \right)$$
(Bertola-Korotkin, 2019)

<u>Prop</u> $\Omega(\Sigma)$ is invariant under transf. $(\Sigma, J) \to (\Sigma', J')$ by moves s.as:
- edge contractions
- merging moves
- attaching / detaching edges to vertices.

Thanks to this theory, we can then compute the two-form ω_k first by seeing it as the standard 2-form associated to a certain graph (Σ, J) (the Stokes graph) and then by appropriately modifying this graph and its connection and introducing in this modification the variables $\{y_i\}$.

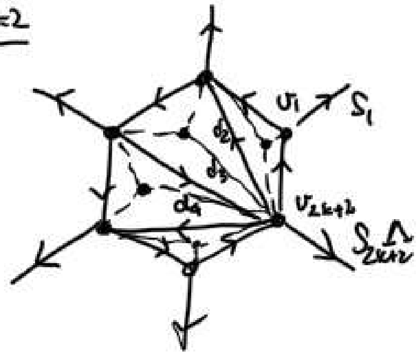
$\to$ Consider the Stokes graph:



By the very definition of $\Omega(\Sigma^{(k)})$ we can say that:
$$\boxed{\omega_k = \frac{1}{2} \Omega\left(\Sigma^{(k)}\right)}.$$

$\to$ Consider the graph as before but that instead of the only vertex v has a $2k+2$-gon with vertices on the Stokes rays and fix a triangulation of this polygon, i.e. consider the graph:

$\underline{K=2}$



where the matrices on the new edges now dep. on some new variables $\{y_\ell\}_{\ell=1}^{2k}$ (and so do the Stokes matrices), in the following way (8)

1) along the perimeter $v_{2k} \to v_{2k+1}$, $v_{2k+2} \to v_1$:

$$D(x_{2k}) = \begin{pmatrix} x_{2k}^{-1} & 0 \\ 0 & x_{2k} \end{pmatrix} \quad k=1\dots K$$

with

$$x_1 = y_1 \prod_{d_j \to v_1} y_j, \quad x_\ell = y_1 \prod_{1\leq k\leq \ell} \prod_{d_j \to v_k} y_j^{(-1)^{k+1}} \quad \ell = 2,..,2K+1, \quad x_{2K+2} = y_1$$

<u>Notice that</u>: for our triang. T_0: $x_1 = y_1 = x_{2K+2}$, $x_\ell = \prod_{j=1}^{\ell} y_j^{(-1)^{j+1}}$ $\ell = 2..2K+1$.

2) along the perimeter edges $v_{2k+1} \to v_{2k+2}$ $k=0..K$

$$V(x_{2k+1}^{-1}) = \begin{pmatrix} 0 & -x_{2k+1}^{-1} \\ x_{2k+1} & 0 \end{pmatrix}$$

3) along the dashed edges inside the triangles

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad (\text{s.t. } A^3 = \mathbb{1}_2).$$

4) along d_j, $j = 2..2K$

$$V(y_j) = \begin{pmatrix} 0 & -y_j \\ y_j^{-1} & 0 \end{pmatrix}.$$

By the general theory, we can do two different things now:

1) Using property 2. of the definition of $\Omega(\Sigma)$, we can now parametrize the Stokes parameters / formal monodromy exponent in terms of the y_j-variables:

$$S_1 = \left(V(y_1^{-1})\, A\, D(y_1)^{-1}\right)^{-1}$$

$$S_2 = \left(D(x_2)\, A\, V(y_2)^{-1}\, A\, V(y_1^{-1})^{-1}\right)^{-1}$$

$$S_{2k} = \left(D(x_{2k})\, A\, V(y_{2k})^{-1}\, A\, V(x_{2k-1}^{-1})^{-1}\right)^{-1} \quad k=2..K$$

$$S_{2k+1} = \left(V(x_{2k+1}^{-1})\, A\, V(y_{2k+1})\, A\, D(x_{2k})^{-1}\right)^{-1} \quad k=1..K-1$$

$$S_{2K+1} = \left(V(x_{2K+1}^{-1})\, A\, D(x_{2K})^{-1}\right)^{-1}$$

$$S_{2K+2}\Lambda = \left(D(y_1) \cdot \prod_{j=2}^{2K} \left(A V(y_j)^{(-1)^j}\right) A\, V(x_{2K+1}^{-1})^{-1}\right)^{-1}.$$

i.e. obtaining the following parametrization:

$$\begin{cases} s_1 = -y_1^{-2} \\ s_{2k} = (1+y_{2k}^2) \prod_{1\le j\le 2k} y_j^{2(-1)^{j+1}} & , k = 1..K \\ s_{2k+1} = -(1+y_{2k+1}^2) \prod_{1\le j \le 2k+1} y_j^{2(-1)^j} & , k = 1..K-1 \\ s_{2K+1} = -\prod_{1\le j\le 2K} y_j^{2(-1)^j} \\ s_{2K+2} = y_1^2 \left(1+y_2^2\left(1+\dots (1+y_{2K}^2)\right)\dots\right) \prod_{j=1}^{K} y_{2j}^{-4} \\ \lambda = (-1)^K \prod_{j=1}^{K} y_{2j}^2 . \end{cases}$$

2) We can compute $\mathcal{W}_K$ in function of $\{y_i\}_{j=1}^{2K}$. Indeed the graph $\Sigma^{(k)}$ is the "total contraction" of the graph $\Sigma_0^{(k)}$, so by exploiting the invariance of the 2-form under canonical moves, we have:

$$\boxed{\mathcal{W}_K = \frac{1}{2}\,\Omega\left(\Sigma^{(k)}\right) = \frac{1}{2}\,\Omega\left(\Sigma_0^{(k)}\right)} .$$

From the last equality one achieve that $\boxed{\mathcal{W}_K = 4 \sum_{\substack{j=1 \\ l\ge j}}^{K} d\log y_{2j-1} \wedge d\log y_{2l}}$

$= d\log \vec{y}\ \Omega_K\, d\log \vec{y}^{\,T}$

This last computation is done just by using the definition and considering the contribution of each vertex in the graph Σ_0, and it's very simple.

Indeed:

- $\rightarrow$ the vertices z_k gives zero contr. since A is const.
- $\rightarrow$ the vertices v_1, v_{2K+1} as well
- $\rightarrow$ the vertices $v_2 .. v_{2K}$ give one contribution each.
- $\rightarrow$ the last vertex has $2K$ nontrivial contrib. to take into account.

Corollary

The induced Poisson structure (by $\mathcal{W}_K$) reads as:

$$\{y_i, y_j\} = P_{ij}\, y_i y_j \qquad (\text{log-canonical since the brackets } \{\log y_i, \log y_j\} = \text{const})$$

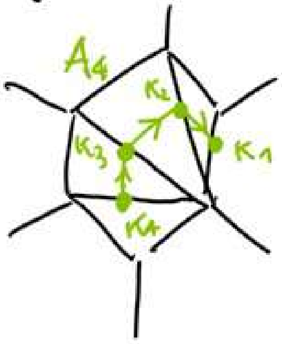
where $P = \Omega_K^{-T}$ and Ω_K is the constant matrix coeff. of the symplectic 2-form $\mathcal{W}_K$ in the coordinates $\{\log y_i\}_{i=1}^{2K}$.

$$P = -\frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & -1 & 0 \end{pmatrix}.$$

Rmk Define $\kappa_\ell = y_\ell^2$ $\ell = 1..2k$ and $B = -4P$: then the variables κ_ℓ are cluster variables of the initial seed $(Q, \vec{\kappa})$ where Q is the Dynkin diagram of type A_{2k} described by adjacency matrix B.
This quiver can be deduced from our construction of $\Sigma_0^{(k)}$ as follows:

- $V(Q)$: $y_1 \dots y_{2k}$ one for each edge of $\Sigma_0^{(k)}$ that depends only on y_i.
- $E(Q) = \{ y_i \to y_j$ every time d_i preceeds d_j turning around their common endpoint in clockwise sense on two edges of the same triangle

E.g. $k=2$, T_0 triangulation:



$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -4P.$$

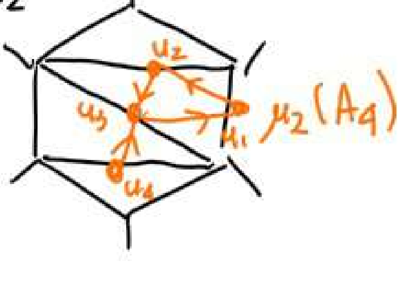
To see that these variables $\kappa_j = y_j^2$ $j = 1..2k$ has indeed cluster algebra structure, one has to look at the other seeds or equivalently to the other triangulations.
Performing a *flip* (Whitehead move) in the triangulation, i.e. exchanging the diagonal in one of the quadrilateral contained in the triangulation T_0 is equivalent to perform a mutation in the quiver A_{2k} (obtained from $\Sigma_0^{(k)}$) w.r.t. the vertex lying on the flipped diagonal of the triangulation.
The result is obtained by comparing the parametrization of the Stokes parameters in the "old" and "new" variables:

$$S_i(\vec{y}) = S_i(\vec{\tilde{y}}) \quad i = 1.. 2k+3$$

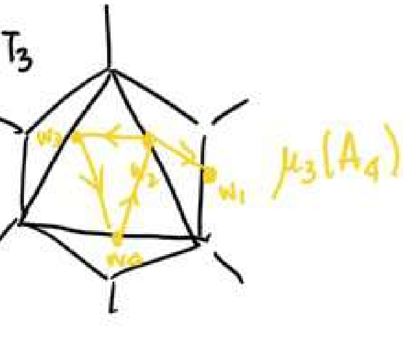
that gives indeed $\vec{\tilde{y}}$ as $\mu_k(\vec{y})$, k being the vertex index of the variable y_k along the flipped diagonal d_k.

E.g. $k=2$

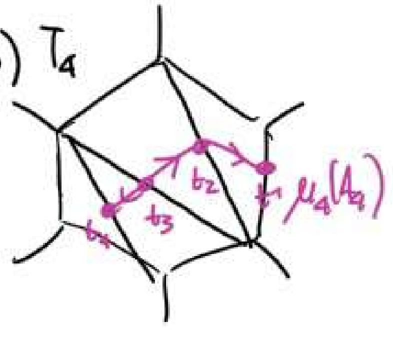
1) T_2



2) T_3



3) T_4



In this case naming

1) $u_1 \ldots u_4$ the variables in T_2, $(\mu_i = u_i^2)$, by imposing:

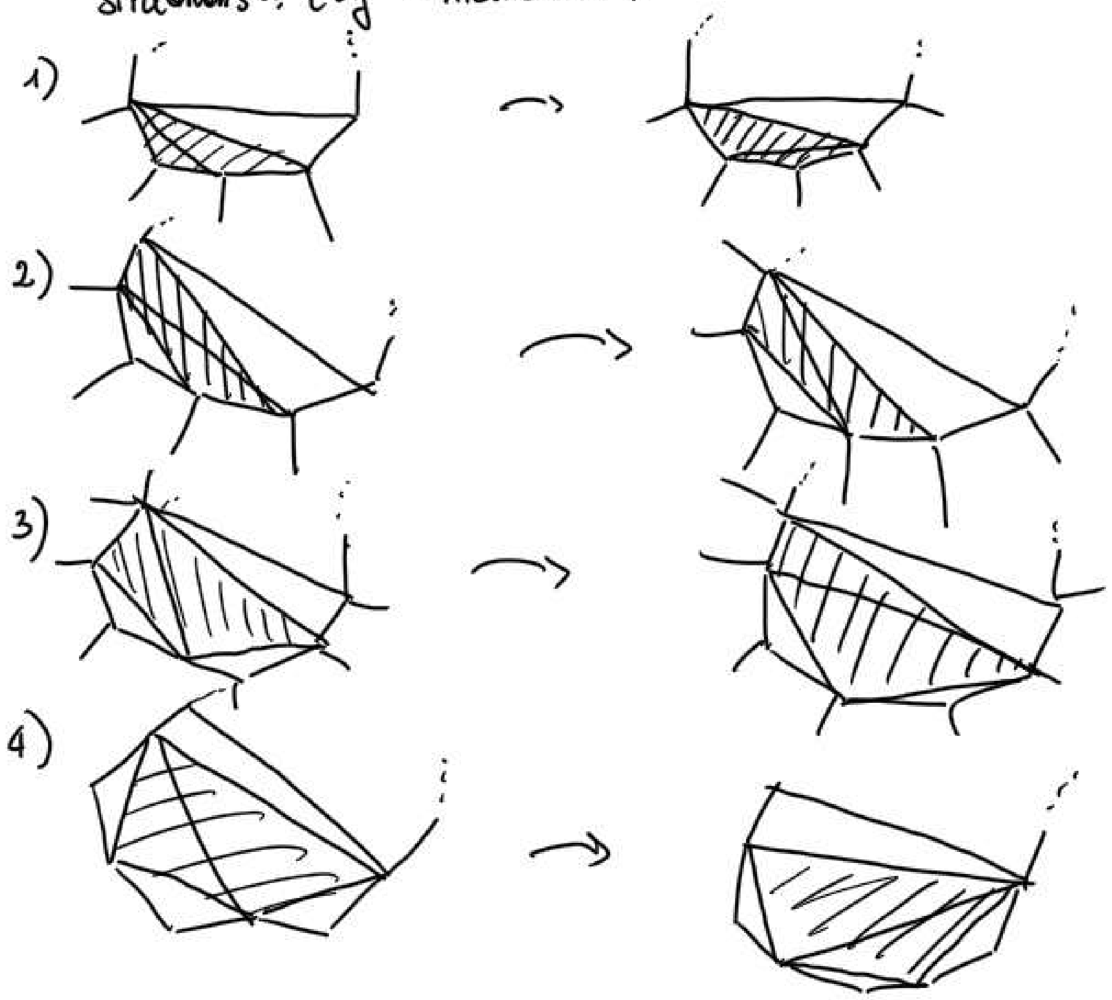
$$S_i(\vec{y}) = S_i(\vec{u}) \quad i = 1 \ldots 7$$

one finds that $\vec{u}^2 = \mu_2(\vec{y}^2)$ " the new variables u_i^2 are obtained by $\mathcal{X}$-cluster mutation relation w.r.t. the vertex y_2 from the initial Dynkin diagram A_4".

2) $w_1 \ldots w_4$, by the same reasoning $\vec{w}^2 = \mu_3(\vec{y}^2)$.

3) $t_1, \ldots t_4$, " " $\vec{t}^2 = \mu_4(\vec{y}^2)$.

__Rmk__ In the general case of the $(2k+2)$-gon one has to check all the possible situations: (by "induction")



Thus the variables $\{\kappa_j = y_j^2\}_{j=1}^{2k}$ are $\mathcal{X}$-cluster variables of A_{2k} type associated to the quiver A_{2k} induced by the graph $\Sigma_0^{(k)}$.

At last, one can finally compute the P. brackets induced by ω_K on the monodromy parameters, using their explicit parametrization in terms of $\{y_j\}_{j=1}^{2k}$.

Prop

The P. brackets induced by ω_K on the monodromy parameters coincide with:

$$\{s_j, s_\ell\}_{FN} = \delta_{j,\ell-1} - \frac{\delta_{j,1}\,\delta_{\ell,2k+2}}{\lambda^2} + (-1)^{j-\ell+1} s_j s_\ell \,, \quad j<\ell$$

$$\{s_j, \lambda\}_{FN} = (-1)^j s_j \lambda$$

the Flaschka-Newell Poisson brackets

Thus the variables $\{y_j\}_{j=1}^{2k}$ "linearize" the F-N. P. brackets.

Rmk The above brackets were defined by F-N in 1981 in a paper called "The inverse monodromy transform is a canonical transformation", where the authors were studying solutions of reductions of the mKdV flows in terms of isomonodromic deformations indeed. They proved that the P. structure that defines the mKdV hierarchy Hamiltonian structure corresponds, at the level of the space of monodromy data, to this Poisson bracket, proving in this way that the "inverse monodromy transform" is a canonical transformation.

~~~

Further questions

1) sl_n case, $n>2$.

In this case $\dim \mathcal{G}_K = n\cdot(n-1)\cdot k$.

Here in the construction the missing variables, further than $\{y_i\}_{j=1}^{2k}$ should be inserted in the $n\times n$ matrices that will replace the const. A: x_{abc} that are $\frac{(n-1)(n-2)}{2}$ in each one of the triangles (that are k).

2) The P_{II} hierarchy case.

In this case one should study $\frac{d\varphi(z)}{dz} = A(z)\varphi(z)$ where

$$A(z) = \sum_{j=0}^{2k} A_j z^j + \frac{\alpha}{z}, \text{ with the symmetry condition } A(z) = -\sigma_1 A(-z)\sigma_1$$

The symmetry condition implies a symmetry in the Stokes parameters ($s_{2k+2-i} = s_i$) that should be reflected in the graph $\Sigma_0^{(k)}$ taking triangulations of $\Sigma_0^{(k)}$ with $\mathbb{Z}_2$ symmetry.

Moreover the presence of the Fuchsian singularity should be properly added and studied.
~~~