Toeplitz determinants related to the discrete Painlevé II hierarchy

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CORTIPOM Workshop

CIRM, Marseille
12 July 2023

Based on a joint work with T. Chouteau SIGMA 19 (2023) 030

Outline

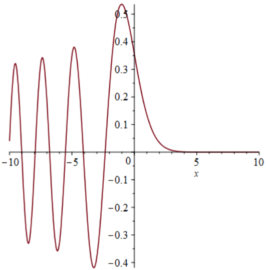
The Airy kernel

The Airy function $\mathrm{Ai}(x)$ is a rapidly decaying at $+\infty$ real solution of the Airy equation

$$v''(x) = xv(x)$$

which can be represented by

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{t^3}{3} + xt\right) dt, \; x \in \mathbb{R}.$$



The Airy kernel $K^{\mathrm{Ai}}(x,y)$ is then built in two equivalent ways

$$K^{\mathrm{Ai}}(x,y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} = \int_0^{+\infty} \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt, \; (x,y) \in \mathbb{R}^2$$

and the integral operator $\mathcal{K}^{\mathrm{Ai}}$ acting on $f \in L^2(\mathbb{R})$ through the Airy kernel acts like

$$\mathcal{K}^{\mathrm{Ai}} f(x) = \int_{\mathbb{R}} K^{\mathrm{Ai}}(x,y) f(y) dy.$$

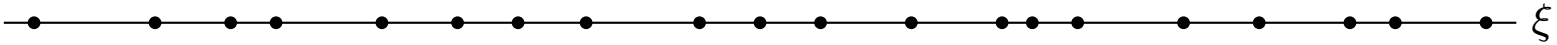
The Airy determinantal point process

[Soshnikov, 2000] Hermitian locally trace class operator $\mathcal{K}$ on $L^2(\mathbb{R})$ defines a determinantal point process on $\mathbb{R}$ if and only if $0 \leq \mathcal{K} \leq 1$. If the corresponding point process exists it is unique.

$$\downarrow$$

The Airy DPP $\mathbb{P}^{\mathrm{Ai}}$ is the point process on $\mathbb{R}$ described by correlation functions

$$\rho(x_1, \ldots, x_k) = \det_{i,j=1,\ldots k} K^{\mathrm{Ai}}(x_i, x_j).$$



Properties Each configuration in $\mathbb{P}^{\mathrm{Ai}}$ counts almost surely an infinite number of points and a largest point. The probability distribution particle of the largest point is given by

$$F(s) := \det\left(1 - \mathcal{K}^{\mathrm{Ai}}|_{(s,+\infty)}\right) = 1 + \sum_{n\geq 1}^{\infty} \frac{(-1)^n}{n!} \int_{(s,\infty)^n} \det_{i,j=1,\ldots n} K^{\mathrm{Ai}}(x_i, x_j) dx_1 \ldots dx_n.$$

The Tracy-Widom formula

[Tracy - Widom, 1994] The Fredholm determinant $F(s)$ satisfies

$$\frac{d^2}{ds^2} \ln F(s) = -u^2(s)$$

where u is the Hastings-McLeod solution of the Painlevé II equation, i.e. the unique solution of the boundary value problem

$$u''(s) = su(s) + 2u^3(s)$$

together with the condition $u(s) \sim \mathrm{Ai}(s)$ for $s \to +\infty$.

Integrating, we have

$$F(s) = \exp\left(-\int_s^{+\infty} (r-s)u^2(r)dr \right).$$

Remark [Picard, 1889 - Painlevé, 1900 - Gambier, 1910] Painlevé equations are 6 nonlinear second order ODEs which do not have movable branch points and generically do not admit solutions in terms of classical special functions.

Higher order Airy kernels

For any $n \geq 1$, we can construct $\mathcal{K}^{\mathrm{Ai}_n}$ the integral operator acting through the n-th order Airy kernel

$$K^{\mathrm{Ai}_n}(x,y) := \int_0^{+\infty} \mathrm{Ai}_n(x+t)\mathrm{Ai}_n(y+t)dt,$$

Ai_n being the n-th Airy function

$$\mathrm{Ai}_n(x) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{t^{2n+1}}{2n+1} + xt\right) dt, \; x \in \mathbb{R}.$$

$\mathcal{K}^{\mathrm{Ai}_n}$ defines a new DPP with largest particle. The Fredholm determinant

$$F_n(s) := \det\left(1 - \mathcal{K}^{\mathrm{Ai}_n}|_{(s,+\infty)}\right)$$

is the distribution function of the largest particle.

Generalization of the Tracy-Widom formula

[Cafasso - Claeys - Girotti, 2019] The Fredholm determinant $F_n(s)$ satisfies

$$\frac{d^2}{ds^2} \ln F_n(s) = -(u^{(n)}((-1)^{n+1}s))^2$$

where $u = u^{(n)}$ solves the n-th member of the homogeneous Painlevé II hierarchy with boundary condition $u(s) \sim Ai_n(s)$ for $s \to +\infty$.

Or again, integrating

$$F_n(s) = \exp\left(-\int_s^{+\infty} (r-s)\left(u^{(n)}((-1)^{n+1}r)\right)^2 dr\right).$$

Remark This result was conjectured in [Le Doussal - Majumdar - Schehr , 2018] where $F_n(s)$ was used to describe the limiting edge behavior of the distribution of momenta of free fermions at zero temperature.

The Painlevé II hierarchy

The Painlevé II hierarchy is a sequence of nonlinear ordinary differential equations

$$\left(\frac{d}{ds}+2u\right)\mathcal{L}_n\left[u_s-u^2\right]=su+\alpha_n,\quad n\geq 1$$

where $\mathcal{L}_n[u_s-u^2]$ are differential polynomilas in u called Lenard polynomials.

They are computed through the following recursion

$$\frac{d}{ds}\mathcal{L}_{n+1}\left[w\right]=\left(\frac{d^3}{ds^3}+4w\frac{d}{ds}+2w_s\right)\mathcal{L}_n\left[w\right],\quad n\geq 0\ \text{ with }\ \mathcal{L}_0\left[w\right]=\frac{1}{2},$$

replacing $w=u_s-u^2$.

Examples

$$\begin{aligned}
n=1: &\qquad u''-2u^3=su+\alpha_1,\\
n=2: &\qquad u''''-10u(u')^2-10u^2u''+6u^5=su+\alpha_2,\\
n=3: &\qquad u''''''-14u^2u''''-56uu'u'''-70(u')^2u''-42u(u'')^2+70u^4u''\\
&\qquad +140u^3(u')^2-20u^7=su+\alpha_3.
\end{aligned}$$

And its Lax pair

[Flaschka - Newell, 1980, Clarkson - Joshi - Mazzocco, 2006] The n-th member of the Painlevé II hierarchy admits the Lax pair representation in terms of the differential 2×2 system

$$\frac{\partial \Psi}{\partial \lambda} = M^{(n)} \Psi, \quad \frac{\partial \Psi}{\partial s} = L \Psi,$$

where the coefficients are

$$L(\lambda, s) = -i\lambda\sigma_3 + u\sigma_1.$$

$$M^{(n)}(\lambda, s) = \left(\sum_{j=0}^{2n} A_j (i\lambda)^j - it \right) \sigma_3 + \sum_{j=0}^{2n-1} \left(B_j \sigma_+ + C_j \sigma_- \right) (i\lambda)^j + \frac{\alpha_n}{\lambda} \sigma_1,$$

with σ_i being the Pauli's matrices, $\sigma_\pm$ are the diagonal elementary matrices and A_j, B_j, C_j that are differential polynomials in u defined through closed formulae involving the Lenard polynomials.

This means that, for every n, the compatibility condition

$$\frac{\partial M^{(n)}}{\partial t} - \frac{\partial L}{\partial \lambda} + M^{(n)} L - L M^{(n)} = 0 \text{ is equivalent to } \mathrm{PII}^{(n)}[\alpha_n].$$

Back to multicritical random partitions

[Okunkov, 2001] On the set of all partitions consider the *Schur* measures

$$\mathbb{P}_{\mathrm{Sc.}}(\lambda) = Z^{-1} s_\lambda \left[\theta_1, \ldots, \theta_n\right]^2 ,$$

where s_λ can be computed as

$$s_\lambda \left[\theta_1, \ldots, \theta_n\right] = \det_{i,j} h_{\lambda_i - i + j} \left[\theta_1, \ldots, \theta_n\right] ,$$

with $\sum_{k \geq 0} h_k z^k = \mathrm{e}^{v(z)}$, $v(z) = \sum_{i=1}^n \frac{\theta_i}{i} z^i$ and $Z = \mathrm{e}^{\sum_{i=1}^n \frac{\theta_i^2}{i}}$.

Remark For $n = 1$ with $\mathbb{P}_{\mathrm{P.Pl.}}(\lambda) = \mathbb{P}_{\mathrm{Sc.}}(\lambda)$ with $\theta_1 = \theta$.

The probability distribution of the first part of such a random partition is given by certain Toeplitz determinants

$$\mathbb{P}_{\mathrm{Sc.}}\left(\lambda_1 \leq k\right) = \mathrm{e}^{-\sum_i^n \hat{\theta}_i^2 / i} D_{k-1}\left(\varphi^{(n)}\left[\hat{\theta}_1, \ldots, \hat{\theta}_n\right]\right) ,$$

where the symbol is

$$\varphi^{(n)}\left[\hat{\theta}_1, \ldots, \hat{\theta}_n\right](z) = \mathrm{e}^{w(z)},\; w(z) = v(z) + v(z^{-1}),\; \theta_i \to \hat{\theta}_i = (-1)^{i+1}\theta_i.$$

Recall Harriet's talk

[Betea–Bouttier–Walsh , 2021] Let

$$\theta_i = (-1)^{i+1} \frac{(n-1)!(n+1)!}{(n-i)!(n+i)!} \theta = (-1)^{i+1} \hat{\theta}_i,$$

then the limiting behavior of the distribution of the first part is described, for certain $b = b(n)$, $d = d(n)$, by

$$\lim_{\theta \to \infty} \mathbb{P}_\theta^n \left(\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2n+1}}} < s \right) = F_n(s).$$

Remark For $n = 1$, it recovers the result of [Baik - Deift - Johannson, 1999], which gave the final answer to the Ulam problem for the limiting behavior of the lenght of the longest increasing subsequence of uniform random permutations.

Outline

Our Toeplitz determinants

Let $\varphi = \varphi^{(n)}[\theta_1, \ldots, \theta_n](z) = \mathrm{e}^{w(z)}$ where

$$w(z) := v(z) + v(z^{-1}), \quad v(z) = \sum_{i=1}^{n} \frac{\theta_i}{i} z^i \; z \in S^1$$

Let $T_k(\varphi)$ being the k-th Toeplitz matrix associated to the symbol $\varphi(z)$

$$T_k(\varphi) = \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-k+1} & \varphi_{-k} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-k+2} & \varphi_{-k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \varphi_{k-1} & \varphi_{k-2} & \cdots & \varphi_0 & \varphi_{-1} \\ \varphi_k & \varphi_{k-1} & \cdots & \varphi_1 & \varphi_0 \end{pmatrix}$$

where for every $h \in \mathbb{Z}$, φ_h is the h-th Fourier coefficient of $\varphi(z)$, namely

$$\varphi_h = \int_{-\pi}^{\pi} e^{-ih\alpha} \varphi(e^{i\alpha}) \frac{d\alpha}{2\pi}, \text{ so that } \sum_{h \in \mathbb{Z}} \varphi_h z^h = \varphi(z).$$

The Toeplitz determinants $D_k = D_k(\varphi)$ are defined as

$$\boxed{D_k := \det(T_k(\varphi))}$$

Borodin formula for $n = 1$

[Borodin, Adler - Van Moerbeke, Baik, 2000] In the case $n = 1$, for every $k \geq 1$ we have

$$\frac{D_k D_{k-2}}{D_{k-1}^2} = 1 - x_k^2$$

where x_k solves the so called *discrete Painlevé II* equation, which corresponds to the second order nonlinear difference equation

$$\theta(x_{k+1} + x_{k-1})(1 - x_k^2) + kx_k = 0$$

with initial conditions $x_0 = -1, x_1 = \varphi_1/\varphi_0$.

Remark Borodin's method is based on the identification of D_k to some quantities related to the discrete Bessel determinantal point process $\rightsquigarrow$ Mattia Cafasso's talk!

The global picture

n=1

Discrete

$\mathbb{P}_{\text{P.Pl.}}(\lambda_1 \le k) = \mathrm{e}^{-\theta^2} D_{k-1}(\varphi)$ with $\varphi = \varphi^{(1)}[\theta_1 = \theta](z)$ and

$$D_{k-2}D_k/D_{k-1}^2 = 1 - x_k^2$$

with x_k solving

$$\text{dPII}: \ \theta(x_{k+1} + x_{k-1})(1 - x_k^2) + kx_k = 0.$$

Continuous

$$\lim_{\theta\to\infty} \mathbb{P}_{\text{P.Pl.}}\left(\frac{\lambda_1 - 2\theta}{\theta^{\frac{1}{3}}} \le s\right) = F(s)$$

and

$$\partial_s^2 \log F(s) = -u^2(s)$$

with u solving

$$\text{PII}: \quad u''(s) = 2u^3(s) + su(s).$$

n>1

$\mathbb{P}_{\text{Sc.}}(\lambda_1 \le k) = \mathrm{e}^{-\sum_i^N \hat{\theta}_i^2/i} D_{k-1}(\varphi)$ with $\varphi = \varphi^{(n)}\left[\hat{\theta}_1, \dots, \hat{\theta}_n\right](z)$ and

what is the recursion relation for D_k?

$\lim_{\theta\to\infty} \mathbb{P}_{\text{Sc.}}\left(\frac{\lambda_1 - b\theta}{(d\theta)^{\frac{1}{2n+1}}} \le s\right) = F_n(s)$ and $\partial_s^2 \log F_n(s) = -u^2((-1)^{n+1}s)$ with u solving the n-th higher order analogue of PII.

Final statement

Theorem (Chouteau - T., 2022)

For any fixed $n \geq 1$, for the Toeplitz determinants $D_k, k \geq 1$, we have

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$

where now x_k solves the $2n$ order nonlinear difference equation

$$kx_k + \left(v_k + v_k Perm_k - 2x_k \Delta^{-1} \left(x_k - (\Delta + I)x_k Perm_k\right)\right) L^n(0) = 0$$

where L is a discrete recursion operator that acts as follows

$$L(u_k) := \left(x_{k+1}\left(2\Delta^{-1} + I\right)\left(\left(\Delta + I\right) x_k Perm_k - x_k\right) + v_{k+1}\left(\Delta + I\right) - x_k x_{k+1}\right) u_k,$$

and $L(0) = \theta_n x_{k+1}$. Here $v_k := 1 - x_k^2$, Δ denotes the difference operator $\Delta : u_k \to u_{k+1} - u_k$ and $Perm_k$ is the transformation

$$\begin{array}{rccc} Perm_k : & \mathbb{C}\left[\left(x_j\right)_{j\in[[0,2k]]}\right] & \longrightarrow & \mathbb{C}\left[\left(x_j\right)_{j\in[[0,2k]]}\right] \\ & P\left(\left(x_{k+j}\right)_{-k\leqslant j\leqslant k}\right) & \longmapsto & P\left(\left(x_{k-j}\right)_{-k\leqslant j\leqslant k}\right). \end{array}$$

The first equations of the hierarchy

$$n=1: \quad kx_k + \theta_1(x_{k+1}+x_{k-1})(1-x_k^2) = 0, \leftarrow$$ discrete Painlevé II equation

$$\begin{aligned} n=2: \quad & kx_k + \theta_1(1-x_k^2)\,(x_{k+1}+x_{k-1}) \\ & + \theta_2(1-x_k^2)\left(x_{k+2}(1-x_{k+1}^2) + x_{k-2}(1-x_{k-1}^2) - x_k(x_{k+1}+x_{k-1})^2\right) = 0, \end{aligned}$$

$$\begin{aligned} n=3: \quad & kx_k + \theta_1(1-x_k^2)\,(x_{k+1}+x_{k-1}) \\ & + \theta_2(1-x_k^2)\left(x_{k+2}(1-x_{k+1}^2) + x_{k-2}(1-x_{k-1}^2) - x_k(x_{k+1}+x_{k-1})^2\right) \\ & + \theta_3(1-x_k^2)\left(x_k^2(x_{k+1}+x_{k-1})^3 + x_{k+3}(1-x_{k+2}^2)(1-x_{k+1}^2) + x_{k-3}(1-x_{k-2}^2)(1-x_{k-1}^2)\right) \\ & + \theta_3(1-x_k^2)\left(-2x_k(x_{k+1}+x_{k-1})(x_{k+2}(1-x_{k+1}^2) + x_{k-2}(1-x_{k-1}^2))\right) \\ & + \theta_3(1-x_k^2)\left(-x_{k-1}x_{k-2}^2(1-x_{k-1}^2) - x_{k+1}x_{k+2}^2(1-x_{k+1}^2)\right) \\ & + \theta_3(1-x_k^2)\,(-x_{k+1}x_{k-1}(x_{k+1}+x_{k-1})) = 0. \end{aligned}$$

Remark Similar discrete equations appeared previously in [Periwal-Schewitz, 1990] in the study of some unitary matrix integrals.

Outline

Orthogonal Polynomials on the Unit Circle

We consider the measure for $z = \mathrm{e}^{i\alpha} \in S^1$ given by

$$\mathrm{d}\mu(\alpha) = \varphi(\mathrm{e}^{i\alpha})\frac{\mathrm{d}\alpha}{2\pi} = \mathrm{e}^{w(\mathrm{e}^{i\alpha})}\frac{\mathrm{d}\alpha}{2\pi}.$$

The family $\{p_k(z)\}_{k\in\mathbb{N}}$ of orthogonal polynomials on the unitary circle (OPUC) w.r.t. the measure is given by

$$p_k(z) = \kappa_k z^k + \ldots \kappa_0, \quad \kappa_k > 0$$

such that the following relation holds for any index k, h

$$\int_{-\pi}^{\pi} \overline{p_k(e^{i\alpha})} p_h(e^{i\alpha}) \frac{\mathrm{d}\mu(\alpha)}{2\pi} = \delta_{k,h}.$$

The analogue monic orthogonal polynomials $\pi_k(z)$ are $p_k(z) = \kappa_k \pi_k(z)$.

Relation with the Toeplitz determinants

A very well known formula of $p_k(z)$ in terms of the Toeplitz determinants D_k gives

$$p_k(z) = \frac{1}{\sqrt{D_k D_{k-1}}} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-k+1} & \varphi_{-k} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-k+2} & \varphi_{-k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{k-1} & \varphi_{k-2} & \cdots & \varphi_0 & \varphi_{-1} \\ 1 & z & \dots & z^{k-1} & z^k \end{pmatrix}, \quad k \geq 1,$$

from which in particular one deduces that the leading coefficient of $p_k(z)$ is related to the ratio of consecutive Toeplitz determinants as

$$\boxed{\frac{D_{k-1}}{D_k} = \kappa_k^2}.$$

Riemann–Hilbert problem associated to OPUC

For any fixed $k \geq 0$, the function $Y(z) := Y(z, k; \theta_i) : \mathbb{C} \to \mathrm{GL}(2, \mathbb{C})$ has the following properties

(1) $Y(z)$ is analytic for every $z \in \mathbb{C} \setminus S^1$;

(2) $Y(z)$ has continuous boundary values $Y_\pm(z)$ are related for all $z \in S^1$ through

$$Y_+(z) = Y_-(z) J_Y(z), \quad \text{with} \quad J_Y(z) = \begin{pmatrix} 1 & z^{-k} e^{w(z)} \\ 0 & 1 \end{pmatrix};$$

(3) $Y(z)$ is normalized at ∞ as

$$Y(z) \sim \left(I + \sum_{j=1}^{\infty} \frac{Y_j(k, \theta_i)}{z^j} \right) z^{k\sigma_3}, \quad z \to \infty,$$

where σ_3 denotes the Pauli's matrix $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Solution of this R–H problem

[Baik - Deift - Johansson, 1999] The R–H problem admits a unique solution $Y(z)$ written as

$$Y(z) = \begin{pmatrix} \pi_k(z) & \mathcal{C}\left(y^{-k}\pi_k(y)\mathrm{e}^{w(y)}\right)(z) \\ -\kappa_{k-1}^2\pi_{k-1}^*(z) & -\kappa_{k-1}^2\mathcal{C}\left(y^{-k}\pi_{k-1}^*(y)\mathrm{e}^{w(k)}\right)(z) \end{pmatrix},$$

where $\pi_{k-1}^*(z)$ is defined as the polynomial of the same degree of $\pi_{k-1}(z)$ such that $\pi_{k-1}^*(z) := z^k\overline{\pi_{k-1}\left(\bar{z}^{-1}\right)}$ and $(\mathcal{C}f(y))(z)$ is the Cauchy transform of f

$$(\mathcal{C}f(y))(z) := \frac{1}{2\pi i}\int_{S^1}\frac{f(y)}{y-z}\mathrm{d}y.$$

Moreover, $\det(Y(z)) \equiv 1$.

Remark This is an extension of the R–H approach to orthogonal polynomials on the real line formulated by Fokas–Its–Kitaev (1991).

Formula for the Toeplitz determinants

For every fixed $k \geq 0$, the unique solution $Y(z)$ of the R–H problem evaluated in $z = 0$ gives

$$Y(0, k; \theta_i) = \begin{pmatrix} x_k & \kappa_k^{-2} \\ -\kappa_{k-1}^2 & x_k \end{pmatrix},$$

with $x_k := \pi_k(0)$ and κ_k the leading coefficient of $p_k(z)$.

Remark Since $\det Y(0, k; \theta_i) = 1$ for every $k \geq 1$ we have

$$\boxed{\frac{\kappa_{k-1}^2}{\kappa_k^2} = 1 - x_k^2}$$

It follows directly that for every $k \geq 1$ we have the recursion for the Toeplitz determinats

$$\frac{D_{k-2} D_k}{D_{k-1}^2} = 1 - x_k^2.$$

The new Lax pair for the dPII hierarchy

Now we construct the function

$$\Psi(z,k;\theta_i) := \begin{pmatrix} 1 & 0 \\ 0 & \kappa_k^{-2} \end{pmatrix} Y(z,k;\theta_j) \begin{pmatrix} 1 & 0 \\ 0 & z^k \end{pmatrix} \mathrm{e}^{w(z)\frac{\sigma_3}{2}}.$$

Now $\Psi(z,k)$ solves the following system

$$\boxed{\Psi(z,k+1) = U(z,k)\Psi(z,k), \quad \partial_z\Psi(z,k) = T(z,k)\Psi(z,k)}$$

with

$$U(z,k) := \begin{pmatrix} z + x_k x_{k+1} & -x_{k+1} \\ -(1-x_{k+1}^2)x_k & 1 - x_{k+1}^2 \end{pmatrix},$$

$$T(z,k) := T_1(k)z^{n-1} + T_2(k)z^{n-2} + \ldots + T_{2n+1}(k)z^{-n-1},$$

where $T_1(n) = \frac{\theta_N}{2}\sigma_3$ and the others $T_\ell(k)$ are obtained in terms of $x_{k-\ell},\ldots,x_{k+\ell}$ explicitely, with x_k solving the n-th equation of the discrete Painlevé II hierarchy.

Remark This comes from the compatibility condition of the system for Ψ, obtained by exchanging the difference and differential operators, namely the condition

$$\sigma_+ = T(z,k+1)U(z,k) - U(z,k)T(z,k).$$

Examples $n = 1, 2$

$\rightsquigarrow$ For $n = 1$ the matrix $T(z, k)$ is

$$T(z,k) = \frac{\theta_1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{z}\begin{pmatrix} n & -\theta_1 x_{k+1}\\ -\theta_1 v_k x_{k-1} & 0\end{pmatrix} + \frac{\theta_1}{z^2}\begin{pmatrix}\frac{1}{2} - x_k^2 & x_k\\ v_k x_k & x_k^2 - \frac{1}{2}\end{pmatrix}$$

and x_k solves the dPII equation.

$\rightsquigarrow$ For $n = 2$ the matrix $T(z, k)$ is

$$\begin{aligned} T(z,k) = {} & z\frac{\theta_2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}\frac{\theta_1}{2} & -\theta_2 x_{k+1}\\ -\theta_2 x_{k-1} v_k & -\frac{\theta_1}{2}\end{pmatrix} + \\ & \frac{1}{z}\begin{pmatrix} k - \theta_2 x_{k-1}x_{k+1}v_k & -\theta_1 x_{k+1} - \theta_2(v_{k+1}x_{k+2} - x_k x_{k+1}^2)\\ \left(-\theta_1 x_{k-1} - \theta_2(v_{k-1}x_{k-2} - x_k x_{k-1}^2)\right) v_k & \theta_2 x_{k-1}x_{k+1}v_k\end{pmatrix} \\ & + \frac{1}{z^2}\begin{pmatrix}-\theta_2 v_k(x_k x_{k-1} + x_k x_{k+1}) + \frac{\theta_1}{2}(v_k - x_k^2) & -\theta_2(v_k x_{k-1} + x_k^2 x_{k+1})\\ -\theta_2(v_k x_{k+1} + x_k^2 x_{k-1})v_k & \theta_2 v_k(x_k x_{k-1} + x_k x_{k+1}) - \frac{\theta_1}{2}(v_k - x_k^2)\end{pmatrix} \\ & + \frac{\theta_2}{z^3}\begin{pmatrix}\frac{1}{2} - x_k^2 & x_k\\ v_k x_k & x_k^2 - \frac{1}{2}\end{pmatrix} \end{aligned}$$

and x_k solves the second equation of the dPII hierarachy.

Here $v_k = 1 - x_k^2$.

Cresswell-Joshi Lax pair

[Cresswell - Joshi, 1998] defined the discrete Painlevé II hierarchy as a sequence of discrete nonlinear equations of order $2n$ for each n for x_k, starting with the discrete Painlevé II and admitting a continuous limit to the classical Painlevé II hierarchy.

They introduced it as the compatibility condition of the system

$$\boxed{\Phi(z,k+1) = L(z,k)\Phi(z,k) \quad \partial_z\Phi(z,k) = M(z,k)\Phi(z,k)}$$

i.e. as the equation

$$\frac{\partial}{\partial z}L(z,k) = M(z,k+1)L(z,k) - L(z,k)M(z,k),$$

where $L(z,k) := \begin{pmatrix} z & x_k \\ x_k & 1/z \end{pmatrix}$ and $M(z,k) := \begin{pmatrix} A_k(z) & B_k(z) \\ C_k(z) & -A_k(z) \end{pmatrix}$ with A_k, B_k and C_k written in integer powers of z (from z^n to z^{-n}) with coefficients depending on x_k.

Remark The relation with the Cresswell-Joshi Lax pair is given by

$$\Phi(z,k) := \sigma_3 \begin{pmatrix} z^{-k+3/2} & 0 \\ 0 & z^{-k+1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_{k-1} & 1 \end{pmatrix} \Psi(z^2, k-1).$$

Outline

The first continuous limit

Main point

The Toeplitz determinants satisfy a recursion relation which is the discrete analogue of the generalized Tracy-Widom formula for the higher order Airy Fredholm determinants.

Recall the B–B–W(=B–D–J) result for $n = 1$: $\lim_{\theta\to\infty} \mathbb{P}_{\text{Sc.}}\left(\frac{\lambda_1 - 2\theta}{\theta^{1/3}} \leq s\right) = F(s)$.
In the limit for $\theta \to \infty$, taking $k = s\theta^{1/3} + 2\theta$ or $s = (k - 2\theta)\theta^{-1/3}$

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2, \qquad x_{k+1} + x_{k-1} = -\frac{k x_k}{\theta(1 - x_k^2)}$$

$$\text{B–B–W}\downarrow \boxed{x_k = (-1)^k \theta^{-1/3} u(s)} \qquad \downarrow \boxed{x_k = (-1)^k \theta^{-1/3} u(s)}$$

$$\partial_s^2 \log F(s) = -u^2(s), \qquad \underbrace{u''(s) = 2u^3(s) + tu(s)}_{\text{Painlevé II equation}}$$

This recovers the Tracy-Widom formula (1994) for $F(s) = \det(1 - \mathcal{K}^{\text{Ai}}|_{(s,+\infty)})$.

The other continuous limits

Recall the B–B–W result for $n = 2$: $\lim_{\theta\to\infty} \mathbb{P}_{\mathrm{Sc.}}\left(\frac{\lambda_1 - \frac{3}{2}\theta}{(4^{-1}\theta)^{1/5}} \le s\right) = F_2(s)$.

In the limit for $\theta \to \infty$, taking $k = s\left(\frac{\theta}{4}\right)^{1/5} + \frac{3}{2}\theta$ (or $s = \left(k - \frac{3}{2}\theta\right)\theta^{-\frac{1}{5}} 4^{\frac{1}{5}}$)

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2,$$

$$k x_k + \theta_1 v_k \left(x_{k+1} + x_{k-1}\right) + \theta_2 v_k \left(x_{k+2} v_{k+1} + x_{k-2} v_{k-1} - x_k (x_{k+1} + x_{k-1})^2\right) = 0$$

B–B–W $\downarrow$ $\boxed{x_k = (-1)^k \left(\frac{\theta}{4}\right)^{-1/5} u(s)}$

$\downarrow$ $\boxed{x_k = (-1)^k \left(\frac{\theta}{4}\right)^{-1/5} u(s) \ \theta_1 = \theta, \ \theta_2 = \frac{\theta}{4}}$

$$\partial_s^2 \log F_2(s) = -u^2(s),$$

$$\underbrace{u'''' - 10u(u')^2 - 10u^2 u'' + 6u^5 = -su}_{\text{2nd eq. of the Painlevé II hierarchy}}$$

which recovers the generalized Tracy-Widom formula for the higher order Airy kernels [Cafasso–Claeys–Girotti, 2019] for $n = 2$. And so on ...

Remark This gives an alternative continuous limit w.r.t. the one proposed by Cresswell – Joshi.

Thank you!