Stokes manifolds and cluster algebras

Sofia Tarricone

LAREMA, Université d'Angers / Concordia University, Montreal

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Based on the join work with M. Bertola ArXiv 2104.13784

The main statement

Theorem (Bertola - T., 2021)

The wild character variety of an $\mathfrak{sl}_2$ polynomial connection of degree K on the Riemann sphere is a cluster manifold of type A_{2K} with one frozen variable.

We consider the algebraic variety of $2K$ complex dimension given by

$$\mathfrak{S}_K = \left\{ \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & s_{2K+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_{2K+2} & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = I_2 \mid s_i \in \mathbb{C}, \ \ \lambda \in \mathbb{C}^* \right\}$$

On $\mathfrak{S}_K$ we define the 2-form

$$\mathcal{W}_K := \frac{1}{2} \sum_{\ell=1}^{2K+3} \mathrm{tr} \left(H_\ell^{-1} \mathrm{d} H_\ell \wedge S_\ell^{-1} \mathrm{d} S_\ell \right), \ \ H_\ell := S_1 \cdots S_\ell, \ \ S_{2K+3} := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \Lambda$$

and $S_{2\ell-1} = \begin{pmatrix} 1 & s_{2\ell-1} \\ 0 & 1 \end{pmatrix}$, $S_{2\ell} = \begin{pmatrix} 1 & 0 \\ s_{2\ell} & 1 \end{pmatrix}$ for $\ell = 1, \dots, K+1$.

1. $(\mathfrak{S}_K, \mathcal{W}_K)$ is symplectic.

The log-canonical coordinates

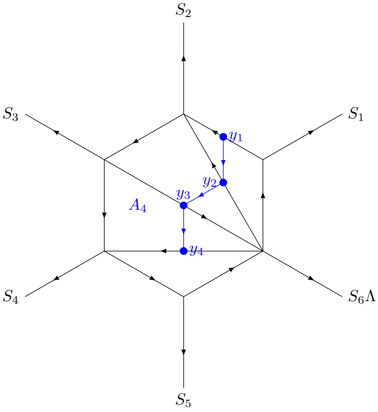
2. We find coordinates $\{y_j\}_{j=1}^{2K}$ on $\mathfrak{S}_K$ that are log-canonical for $\mathcal{W}_K$.

Theorem (Bertola - T., 2021)

The Poisson bracket induced by $\mathcal{W}_K$ is

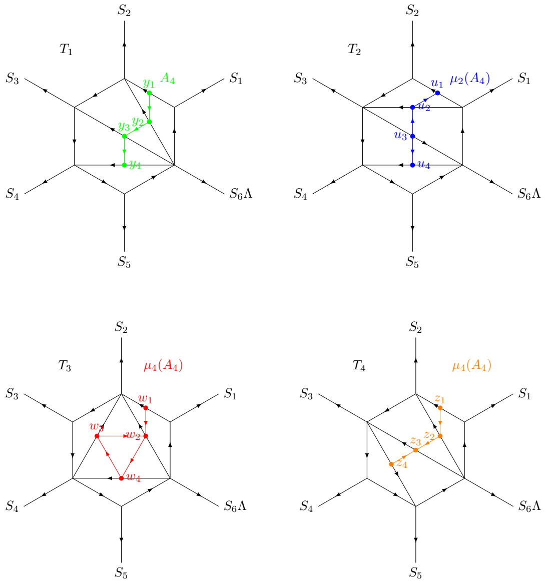
$$\{y_i, y_j\} = \mathbf{P}_K^{ij} y_i y_j \quad \textit{where}$$

$$\mathbf{P}_K = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & & -1 & 0 & 1 \\ 0 & 0 & \dots & & 0 & -1 & 0 \end{pmatrix}$$



3. $4\mathbf{P}_K$ is the adjacency matrix of an A_{2K}-Dynkin diagram.

The cluster relations



4. The flip of each internal edge of the triangulation T_1 corresponds to a Y-mutation in the quiver A_{2K}.

Thank you!